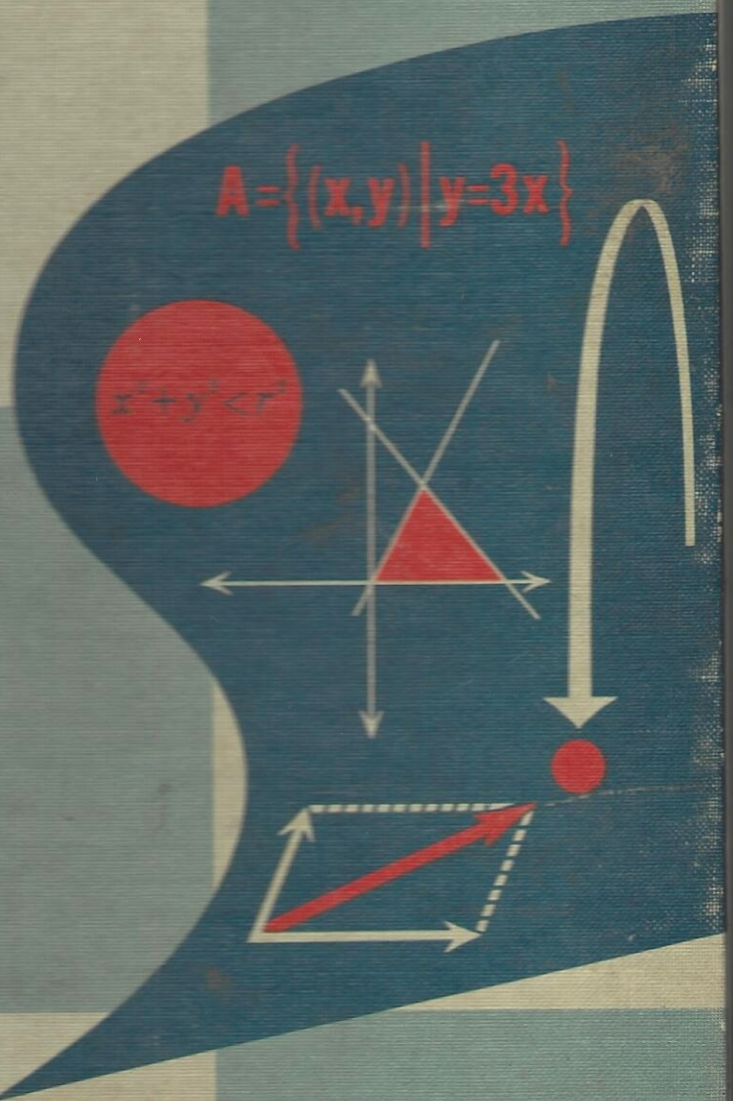
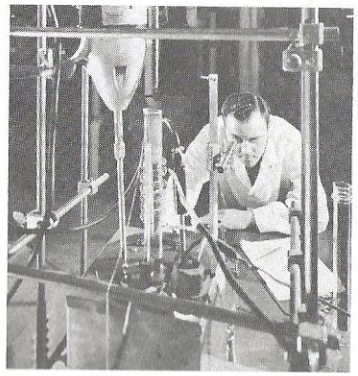
MODERN MATHEMATICS



ALGEBRA ONE

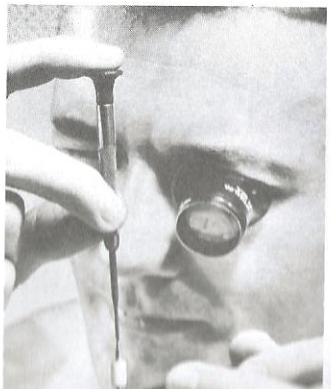
Part 2 of 3

MODERN MATHEMATICS *Algebra One*



Thus,

$$1+4+7+\cdots+(3k-2)+(3k+1)$$
$$=\tfrac{1}{2}k(3k-1)+(3k+1)$$
$$=\tfrac{3}{2}k^2-\tfrac{1}{2}k+3k+1$$
$$=\tfrac{3}{2}k^2+\tfrac{5}{2}k+1$$
$$=\tfrac{1}{2}(3k^2+5k+2)$$
$$=\tfrac{1}{2}(k+1)(3k+2)$$



MODERN MATHEMATICS

Algebra One

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Relating Rational and Irrational Numbers

Preface

TEACHING of elementary algebra today must reflect recent developments in the realm of mathematics. Clearly, one aim of an algebra course is to acquaint the student with the algebra that is used functionally by present-day mathematicians. Therefore, the course must include topics not previously in the elementary algebra curriculum, for example: set theory; structure of number systems; inequalities as well as equations; the function concept; and greater attention to the important role of the coordinate plane. Needless to say, the development of these topics in a beginning course in algebra should be appropriate to the mathematical maturity of the student.

It should be emphasized that topics of modern algebra are not "substitutions" for topics of traditional algebra. Rather, modern topics provide a more meaningful way of developing understandings basic to the traditional topics.

Of crucial importance in the course is the spirit and technique used to bring the content to the student. By stressing individual inquiry and participation, and by leading the student through a carefully planned series of discoveries, teacher and text can provide important benefits for the student. These benefits include an *interest* in mathematics; an *understanding* of the foundations of mathematics; and *manipulative skills* based on understanding.

Algebra is not simply a small mathematical world in itself. It has a far-reaching role throughout mathematics, including the important areas of geometry, trigonometry, and analytic geometry. It is an important function of an elementary algebra course to communicate an awareness of this role to the student.

It is worth observing that modern algebra is not "difficult" algebra, suitable only for the better student. Through a heightened emphasis on meaning and structure, students of average or less-than-average ability can benefit as much as the superior student.

MODERN MATHEMATICS *Algebra One* is carefully planned to reflect the algebra course described above. It is the belief of the authors that teacher and student alike will find this course in modern algebra stimulating, meaningful, and an effective preparation for subsequent courses in mathematics.

CONTINUING FROM

PART ONE

OF THIS BOOK

5 *Rational Expressions*

PROPERTIES

Definition In Chapter 2 we found that the set of integers is not closed under division; for example, $7 \div 5 = \frac{7}{5}$. In order to have a set of numbers that is closed under division, we considered another set of numbers called the rationals. As you know, a rational number may be expressed as the quotient of two integers. Is $\frac{7}{5}$ a rational number? Is the set of integers a subset of the set of rational numbers?

1. Do you think the set of polynomials is closed under division? Is y a polynomial? Is 3 a polynomial? Turn to page 123 and look at **2.** Is the indicated quotient $\frac{3}{y}$ a polynomial? Then we are in somewhat the same position with the set of polynomials as we were with the set of integers, and our actions will be much the same.

Let us construct a new set of expressions that has the polynomials as a subset, but let this set be closed under division. We will call these expressions the set of rational expressions.

A **rational expression** is an expression in the form $\frac{P_1}{P_2}$ where P_1 is a polynomial and P_2 is a polynomial that is not zero. P_1 is the numerator of the expression, and P_2 is the denominator. P_1 and P_2 are the *terms* of the expression.

2. Do the integers belong to the set of polynomials? Then does the integer 1 qualify as a replacement for P_2 in the above definition? Is $\frac{P_1}{1}$ a rational expression? Can $\frac{P_1}{1}$ be expressed as P_1? Do you see that polynomials are a subset of the set of rational expressions? Is it also true that all rational expressions are polynomials? Which of the following expressions are polynomials? Which are rational expressions?

a. $\frac{x+2}{x}$ **b.** $\frac{y}{3}$ **c.** $\frac{x^2+3x+1}{2x-4}$ **d.** x^2 **e.** $\frac{3}{2x}$

3. You know that the denominator in a rational expression may never be zero. In expressions such as $\frac{3}{x}$ you can easily see that the possible replacements for the variable are restricted to numbers other than zero. What restrictions apply to the replacement set for y in the expression $\frac{y}{y+3}$? $\frac{4-y}{3y}$?

Sometimes it is more difficult to determine the possible replacement set for the variable in an expression. Let us examine the expression $\frac{x^2+2}{x^2+5x+6}$. Can you determine the restrictions on x by inspection? Suppose you factor the denominator: $x^2+5x+6=(x+3)(x+2)$. If x is replaced by -3, what will the factor $x+3$ equal? Then what will the denominator be? Suppose x is replaced by -2. What will the denominator be? Why?

Describe the restrictions on the replacement sets for the variables in the following expressions.

a. $\frac{a}{4}$ b. $\frac{a}{b}$ c. $\frac{5}{7y}$

d. $\frac{p+1}{p-1}$ e. $\frac{1}{3x-6}$ f. $\frac{c^2-5}{2c^2-6c-20}$

g. $\frac{x^2+1}{x^2-1}$ h. $\frac{4m^2-2}{3-m}$ i. $\frac{s^2+7s-8}{2s^2+17s+35}$

Reducing Rational Expressions

1. In your work with rational numbers, you often find it convenient to reduce a fraction to lowest terms. May $\frac{15}{20}$ be expressed as $\frac{5}{5}\cdot\frac{3}{4}$? Is $\frac{5}{5}$ another name for the identity element for multiplication? Then $\frac{5}{5}\cdot\frac{3}{4}$ equals $1\cdot\frac{3}{4}$, or $\frac{3}{4}$.

2. You know that the multiplication of rational numbers is defined as $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$ $(b\neq 0, d\neq 0)$. Does this definition make it possible to reduce fractions to lowest terms; that is, to a form in which the only common factor is 1? The reduction is accomplished by factoring the numerator and the denominator and then eliminating all common factors.

3. Use the multiplication definition, factor, and use the identity element for multiplication to reduce the rational numbers below to lowest terms.

a. $\frac{12}{24}$ b. $\frac{-14}{35}$ c. $\frac{54}{72}$ d. $\frac{90}{132}$

Now let us define the multiplication of rational expressions to see how the definition enables us to reduce a rational expression to lowest terms.

The product of two rational expressions, $\frac{P_1}{P_2}$ and $\frac{Q_1}{Q_2}$ (P_2 and $Q_2\neq 0$), is defined as $\frac{P_1}{P_2}\cdot\frac{Q_1}{Q_2}=\frac{P_1Q_1}{P_2Q_2}$.

4. To reduce a rational expression to lowest terms, we use this definition in the same fashion in which we used the definition of multiplication to reduce rational numbers. For example, let us reduce the rational expression $\frac{ax}{ay}$. By the definition, this product is equal to $\frac{a}{a}\cdot\frac{x}{y}$. What does $\frac{a}{a}$ equal? Then is it true that $\frac{a}{a}\cdot\frac{x}{y}$ equals $1\cdot\frac{x}{y}$? Is $\frac{ax}{ay}$ equal to $\frac{x}{y}$? Do the terms of $\frac{x}{y}$ have any common factors other than 1?

5. Test the equation $\frac{ax}{ay}=\frac{x}{y}$ by replacing a with 1, x with 2, and y with 3. Can any number be a member of the replacement set for a? for x? for y? Read the definition of rational expressions again to be sure of your answer.

6. Look at the expression $\frac{ax^2}{xy}$. Can the numerator and denominator of this rational expression be factored as $\frac{a\cdot x\cdot x}{x\cdot y}$? Is x a factor common to the numerator and the denominator? Does $\frac{x}{x}=1$? Then does the expression $\frac{a\cdot x\cdot x}{x\cdot y}$ equal $1\cdot\frac{ax}{y}$? Do $\frac{ax^2}{xy}$ and $\frac{ax}{y}$ represent equal rational expressions?

7. Test the equation $\frac{ax^2}{xy}=\frac{ax}{y}$ by replacing a with 2, x with 3, and y with 4. What are the restrictions on the replacement set for a? for x? for y?

8. The expression $\frac{5x^2y}{10xy^2}$ is reduced by taking the steps shown below. Is the commutative principle applied at any time? the associative? Why may we use these principles on the terms of the expression? What is the replacement set for x? for y?

$$\frac{5x^2y}{10xy^2} = \frac{5 \cdot x \cdot x \cdot y}{2 \cdot 5 \cdot x \cdot y \cdot y}$$
$$= \frac{5 \cdot x \cdot y}{5 \cdot x \cdot y} \cdot \frac{x}{2y}$$
$$= 1 \cdot \frac{x}{2y}, \text{ or } \frac{x}{2y}$$

Reduce the following rational expressions by the method just described.

	a	b	c
9.	$\frac{4x^2}{20x}$	$\frac{16a}{12a}$	$\frac{15bc}{10ac}$
10.	$\frac{5sr}{35s^2r}$	$\frac{-9ab^2c}{15abc^2}$	$\frac{-25xyz}{-30x^2yz}$
11.	$\frac{3x^2}{9x^2}$	$\frac{6x}{3(x+2)}$	$\frac{7(x^2y)^3}{14(xy^2)^3}$

12. Consider the expression in **8** again. After $\frac{5x^2y}{10xy^2}$ is expressed in the factored form $\frac{5 \cdot x \cdot x \cdot y}{2 \cdot 5 \cdot x \cdot y \cdot y}$, it is easy to determine which factors are common to both the numerator and the denominator. Since these common factors will result in a factor of 1 for the rational expression, it is customary to indicate this by placing a slash mark (/) through the common factors and to proceed directly to the simplified result. Your work would then look like this:

$$\frac{5x^2y}{10xy^2} = \frac{\cancel{5} \cdot \cancel{x} \cdot x \cdot \cancel{y}}{2 \cdot \cancel{5} \cdot \cancel{x} \cdot \cancel{y} \cdot y}$$
$$= \frac{x}{2y}$$

Another illustration of using slash marks to indicate the factors common to both terms is:

$$\frac{6ay}{18a^2y} = \frac{\cancel{2} \cdot \cancel{3} \cdot \cancel{a} \cdot \cancel{y}}{\cancel{2} \cdot \cancel{3} \cdot 3 \cdot \cancel{a} \cdot a \cdot \cancel{y}} = \frac{1}{3a}$$

Explain why 1 is the numerator of the final rational expression.

13. This short-cut procedure for reducing expressions to lowest terms applies equally well to more complicated rational expressions. Be sure, however, to factor both the numerator and the denominator and to mark with slash marks only *factors* that are common to both. For example, examine the expression $\frac{a^2-4}{2a+4}$. This can be reduced by the following procedure.

$$\frac{a^2-4}{2a+4} = \frac{(a+2)(a-2)}{2(a+2)}$$
$$= \frac{(a+2)}{(a+2)} \cdot \frac{(a-2)}{2}, \text{ or } \frac{a-2}{2}$$

14. You may shorten the work in **13** to:

$$\frac{a^2-4}{2a+4} = \frac{\overset{1}{\cancel{(a+2)}}(a-2)}{2\underset{1}{\cancel{(a+2)}}} = \frac{a-2}{2}$$

Test this statement by replacing a with 3. Try letting a equal 2. Could -2 be used to replace a? Why not? Look at the example in **12**. By writing "1" as above to show the factor that results when removing common factors from the two terms, you will not overlook a final term of 1 in either the numerator or denominator.

EXERCISES

Reduce the following rational expressions to lowest terms.

1.	$\frac{6}{2x+2y}$	**2.** $\frac{6}{3x+3}$	**3.** $\frac{7a+7b}{7}$
4.	$\frac{m^2-9}{3m-9}$	**5.** $\frac{5a+5b}{5a-5b}$	**6.** $\frac{ax-bx}{ax+bx}$
7.	$\frac{3x+3y}{x^2-y^2}$	**8.** $\frac{x-5}{x^2-6x+5}$	**9.** $\frac{x^2-25}{x^2+6x+5}$
10.	$\frac{2z^2-2}{z+1}$	**11.** $\frac{y^2-25}{3y-15}$	**12.** $\frac{y^2-2y+1}{y^2+5y-6}$
13.	$\frac{2x^2+3x+1}{4x^2-1}$	**14.** $\frac{x^2-2x-3}{2x^2-5x-3}$	**15.** $\frac{2x^2-4xy+2y^2}{(x-y)^2}$
16.	$\frac{4r^2s-s}{4r^2-4r+1}$	**17.** $\frac{x^2+y^2}{x^4-y^4}$	**18.** $\frac{t^4-1}{t^4-2t^2+1}$

Additive Inverses in Rational Expressions

1. It is sometimes desirable to change the form of a rational expression so that it can be reduced to lower terms. Suppose we wish to reduce an expression such as $\frac{3-a}{a-3}$. You know that the numerator $3-a$ may be expressed as $-1(a-3)$, or $-(a-3)$. Then can $\frac{3-a}{a-3}$ be rewritten as $\frac{-(a-3)}{a-3}$? Is the following equation true?

$$\frac{3-a}{a-3} = \frac{-1(a-3)}{a-3} = \frac{-1}{1}, \text{ or } -1$$

2. The polynomials $a-3$ and $3-a$ are additive inverses. Look at the pairs of non-zero additive inverses listed below. In each pair, what is the quotient of the first expression divided by the second? of the second divided by the first?

a. $-3,\ 3$ **b.** $\frac{1}{2},\ -\frac{1}{2}$ **c.** $x^2,\ -x^2$ **d.** $y+3,\ -(y+3)$

Do you think the rational expression formed by the indicated quotient of non-zero opposites always equals -1? We can make the following generalization.

$$\frac{a-b}{b-a} = -1 \quad (a \neq b)$$

3. Study the procedure below used to reduce an expression. At what step is -1 understood to be a factor of the numerator? Since $-1 = \frac{1}{-1} = \frac{-1}{1}$, the result can be written in the three ways shown.

$$\frac{25-x^2}{x^2-6x+5} = \frac{(5-x)(5+x)}{(x-5)(x-1)}$$

$$= \frac{-(x-5)(5+x)}{(x-5)(x-1)} \quad (\text{denominator } 1)$$

$$= \frac{-x-5}{x-1}, \text{ or } \frac{x+5}{-x+1}, \text{ or } -\frac{x+5}{x-1}$$

4. Select pairs of polynomials that are additive inverses and tell what the quotient is for each such pair.

a. $3a-2,\ 3a+2,\ 2-3a,\ -2-3a$

b. $x^2+x-1,\ x^2-x+1,\ -x^2-x+1,\ -x^2+x-1$

EXERCISES

Reduce to lowest terms and express answers in a form that has the fewest minus or negative signs. In each exercise, list any restrictions that apply to the possible replacements for the variable.

1. $\frac{2a-2b}{b-a}$ **2.** $\frac{2b-2c}{c^2-b^2}$ **3.** $\frac{6b-x}{2x-12b}$

4. $\frac{4-9x^2}{12x-8}$ **5.** $\frac{m^2-2m+1}{1-m^2}$ **6.** $\frac{x^2-5x-6}{1-x^2}$

7. $\frac{ax^2-ay^2}{by-bx}$ **8.** $\frac{6-x-x^2}{x^2-x-6}$ **9.** $\frac{16-x^2}{x^2+x-12}$

10. $\frac{2x^2+x-1}{1-4x^2}$ **11.** $\frac{3+a}{-3-a}$ **12.** $\frac{-x^2}{x^3}$

Just for Fun

Below is a "proof" of the impossible relationship $2=1$. See whether you can find the step in the "proof" that is improper. As a clue, we suggest you keep in mind the fact that division by zero is not permitted. Also, does zero have a multiplicative inverse?

Suppose a and b are two names for the same number. Then:

$$a = b$$

(1) Now multiply both members by a: $a^2 = ab$

(2) Add $-b^2$ to both members: $a^2 - b^2 = ab - b^2$

(3) Factor both members: $(a+b)(a-b) = b(a-b)$

(4) Then multiply each member by $\frac{1}{a-b}$: $\frac{1}{a-b} \cdot (a+b)(a-b) = \frac{1}{a-b} \cdot b(a-b)$

$$a + b = b$$

(5) But since $a=b$, we may write: $2b = b$

(6) Now multiply both members by $\frac{1}{b}$: $2 = 1$

Did you discover the improper step?

OPERATIONS WITH RATIONAL EXPRESSIONS

Multiplication

1. You know that the multiplication of rational expressions is defined by:

$$\frac{P_1}{P_2} \cdot \frac{Q_1}{Q_2} = \frac{P_1Q_1}{P_2Q_2} \quad (P_2 \text{ and } Q_2 \neq 0)$$

Is P_1Q_1 a polynomial? Is P_2Q_2? Then is $\frac{P_1Q_1}{P_2Q_2}$ a rational expression? Do you think the set of rational expressions is closed under multiplication?

2. Study the multiplication example below. Explain each step.

$$\frac{3x^2}{8xy} \cdot \frac{4y^2}{6x} = \frac{3x^2 \cdot 4y^2}{8xy \cdot 6x}$$

$$= \frac{\overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{4}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{y}} \cdot y}{\underset{1}{\cancel{2} \cdot \cancel{4}} \cdot 2 \cdot \underset{1}{\cancel{3}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{y}}}$$

$$= \frac{y}{4}$$

3. All rational expressions should be expressed in reduced form unless there is a definite reason for not doing so.

Examine this multiplication example:

$$\frac{x+2}{x^2-4} \cdot \frac{3x-6}{9} = \frac{\overset{1}{\cancel{(x+2)}} \cdot \overset{1}{3} \cdot \overset{1}{\cancel{(x-2)}}}{\underset{1}{\cancel{(x+2)}} \cdot \underset{1}{\cancel{(x-2)}} \cdot 3}$$

$$= \frac{1}{3}$$

The same example is repeated below. However, in this case we have not factored the numerator or denominator or removed common factors before the final product is obtained.

$$\frac{x+2}{x^2-4} \cdot \frac{3x-6}{9} = \frac{(x+2)(3x-6)}{(x^2-4)9}$$

$$= \frac{3x^2-12}{9x^2-36}$$

Do you see that the reduced form of the product is more easily obtained by expressing the numerator and denominator in factored form and removing common factors *before* the product is formed?

4. Test the correctness of the multiplication in **2** by replacing x by 1, y by 2. Test by replacing x by 0, y by 1. Is this possible? What number or numbers are not members of the replacement set for x? Why?

5. Find each pair of products.

a. $\frac{24}{3x-6} \cdot \frac{x^2-2}{4}$ $\qquad$ $\frac{x^2-2}{4} \cdot \frac{24}{3x-6}$

b. $\frac{9a+9b}{3a^2+6ab+3b^2} \cdot \frac{2a-2b}{4a^2-16b^2}$ $\qquad$ $\frac{2a-2b}{4a^2-16b^2} \cdot \frac{9a+9b}{3a^2+6ab+3b^2}$

Do you think that the multiplication of rational expressions is commutative?

6. Find each pair of products.

a. $\left(\frac{r-s}{2r} \cdot \frac{3s}{r+1}\right) \cdot \frac{rs}{4-r}$ $\qquad$ $\frac{r-s}{2r} \cdot \left(\frac{3s}{r+1} \cdot \frac{rs}{4-r}\right)$

b. $\left(\frac{x^2-3}{x^2+1} \cdot \frac{x^2-3x+2}{5}\right) \cdot \frac{4-x^2}{x^2+4}$ $\qquad$ $\frac{x^2-3}{x^2+1} \cdot \left(\frac{x^2-3x+2}{5} \cdot \frac{4-x^2}{x^2+4}\right)$

Does it seem that multiplication of rational expressions is associative?

EXERCISES

Find the products. Express each in reduced form. In each example, give any restrictions on the replacement set for the variables.

	a	b
1.	$\frac{2ab^2}{xy} \cdot \frac{x^2y^2}{4ab}$	$\frac{3x}{x+1} \cdot \frac{2x+2}{x^2}$
2.	$\frac{3x^2-6x}{2x} \cdot \frac{4}{2x-4}$	$\frac{x^2-9}{y^2-4} \cdot \frac{y+2}{x-3}$
3.	$\frac{x^3}{x^2-x-6} \cdot \frac{x^2-1}{x^2}$	$\frac{12x+6}{4x^2-1} \cdot \frac{6x-3}{12}$
4.	$\frac{y^2-2y+1}{7y^2-7} \cdot \frac{14y}{y-1}$	$\frac{x+3}{2} \cdot \frac{3}{x}$

5. Study the multiplication example below. Is it correct? What does $\frac{a-2}{2-a}$ equal?

$$\frac{a^2-4}{a^2}\cdot\frac{a^3}{6-3a}=\frac{(a+2)\overset{-1}{\cancel{(a-2)}}}{\underset{1}{\cancel{a^2}}\cdot 3\underset{1}{\cancel{(2-a)}}}\cdot\frac{\overset{a}{\cancel{a^3}}}{}=\frac{-a(a+2)}{3}$$

Multiply. Express each product in reduced form.

	a	b
6.	$\frac{x^2-y^2}{18}\cdot\frac{9}{y-x}$	$\frac{a^2-b^2}{c}\cdot\frac{1}{b+a}$
7.	$\frac{2b-2a}{z}\cdot\frac{z^2}{a-b}$	$\frac{3c-5d}{25d^2-9c^2}\cdot\frac{5d+3c}{c}$
8.	$\frac{x^2-2x-24}{x+4}\cdot\frac{2}{2x-12}$	$\frac{m^2+10m+25}{m-5}\cdot\frac{m}{m+5}$
9.	$\frac{2ax}{3(x+1)}\cdot\frac{10a}{5x}\cdot\frac{6x+6}{2a^2}$	$\frac{3x^2-3}{x^2-2x+1}\cdot\frac{5x-5}{x^2}\cdot\frac{x}{5x+5}$
10.	$\frac{a-b}{a+b}\cdot\frac{b+a}{b-a}\cdot\frac{-a+b}{a-b}\cdot\frac{a^2-b^2}{a^2+b^2}$	$\frac{m+2n}{m-2n}\cdot\frac{2n-m}{2n+m}$
11.	$\frac{3x^2+7x+2}{4-x^2}\cdot\frac{3x^2-7x+2}{1-9x^2}$	$\frac{x^3-x^2-6x}{4x^2-25}\cdot\frac{2x-10x}{x^2}$

Division

1. You know that the operation of division is defined by means of the multiplicative inverse. Multiplication and division are inverse operations. It is necessary to have a multiplicative identity element before multiplicative inverses can be found. In the set of polynomials, what is the identity element for multiplication: $__?__\cdot P = P$?

What element will satisfy this equation: $__?__\cdot\frac{P_1}{P_2}=\frac{P_1}{P_2}$? Is the number 1 a polynomial? a rational expression? Then what is the multiplicative identity element for the set of rational expressions?

2. If A is an element of a set, the multiplicative inverse of A (written $\frac{1}{A}$) is an element such that the product $\frac{1}{A}\cdot A$ is 1. In the set of rational numbers, what is the multiplicative inverse of 3? of $\frac{2}{3}$? of $\frac{-4}{5}$? of 1? of -1? Is *reciprocal* another name for *multiplicative inverse*?

3. What is the multiplicative inverse of x? of $2x^2$? of $x-1$? of $3x+2$? Are these multiplicative inverses polynomials? Are they rational expressions? Do you need to place a restriction on the replacement set for x in each of these examples?

4. Is $\frac{x-3}{x+8}\cdot\frac{x+8}{x-3}=1$ a true sentence? For what values of x is the sentence not true? May we say that $\frac{x+8}{x-3}$ is the multiplicative inverse of $\frac{x-3}{x+8}$? Is $\frac{x-3}{x+8}$ the multiplicative inverse of $\frac{x+8}{x-3}$?

5. What is the multiplicative inverse of $\frac{x^2-2}{x+7}$? of $\frac{a^2-b^2}{2ab}$? of $\frac{x^2-3xy+2y^2}{x^2-y^2}$?

6. You know that $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$ if b, c, and d do not equal zero. Then if P_1, P_2, P_3, and P_4 are polynomials, we may define division with rational expressions by:

$$\frac{P_1}{P_2}\div\frac{P_3}{P_4}=\frac{P_1}{P_2}\cdot\frac{P_4}{P_3}\quad(P_2,\ P_3,\ P_4\neq 0)$$

For example, $\frac{x}{2}\div\frac{x^2-2}{x+2}=\frac{x}{2}\cdot\frac{x+2}{x^2-2}$, and the quotient is $\frac{x(x+2)}{2(x^2-2)}$. Can this be reduced to lower terms?

Another example of division with rational expressions is:

$$\frac{a+b}{4a}\div\frac{a^2-b^2}{2ab}=\frac{a+b}{4a}\cdot\frac{2ab}{a^2-b^2}$$

$$=\frac{\overset{1}{\cancel{a+b}}}{\underset{2}{\cancel{4a}}}\cdot\frac{\overset{1}{\cancel{2ab}}}{\underset{1}{\cancel{(a+b)}}(a-b)}$$

$$=\frac{b}{2(a-b)}$$

7. Is the set of rational expressions a well-defined set? Define it. Let us see whether we can consider the set of rational expressions (except 0) a commutative group under multiplication.

(1) Is the set closed under multiplication? Give an example.
(2) Is multiplication of rational expressions commutative? Illustrate your answer.
(3) Is multiplication of rational expressions associative? Illustrate your answer.
(4) Is there an identity element for multiplication in the set? What is it?
(5) Is there a multiplicative inverse in the set for every rational expression (except 0)? Give an example.

We will assume that the set of rational expressions is a commutative group under multiplication.

EXERCISES

Divide. Express quotients in reduced form.

1. $\frac{ab}{bc} \div \frac{ad}{db}$
2. $\frac{2x^2y}{15xy^2} \div \frac{5x^3y}{4xy}$
3. $\frac{3a-6b}{a^2-4b^2} \div \frac{6a}{4a+8b}$
4. $\frac{x}{x^2-4x+4} \div \frac{x^2}{x^2-4}$
5. $\frac{3x^2y^2-2xy-1}{x^2y^2-1} \div \frac{3xy-1}{xy+1}$
6. $\frac{2y^2-50}{5y^2-80} \div \frac{3y-15}{4y+16}$

Perform the indicated operations.

7. $\left(\frac{3xy}{6xy^2} \cdot \frac{x^2}{y}\right) \div \frac{9y}{4x}$
8. $\frac{3xy}{6xy^2} \cdot \left(\frac{x^2}{y} \div \frac{9y}{4x}\right)$
9. $\left(\frac{a+b}{a-b} \cdot \frac{a^2-b^2}{3a-3b}\right) \div \frac{5b+5a}{2b-2a}$
10. $\frac{5x^2-6x+1}{1+x} \cdot \left(\frac{x+1}{25x^2-1} \div \frac{x^2-1}{10x+2}\right)$

Addition

1. You know that addition of rational numbers is defined by the statement $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$, where a, b, and c are integers and $b \neq 0$. The addition of rational expressions is defined in the same way.

▶ The definition of addition for rational expressions is:

$\frac{P_1}{P_2} + \frac{P_3}{P_2} = \frac{P_1+P_3}{P_2}$, where P_1, P_2, and P_3 are polynomials and $P_2 \neq 0$

2. Consider the addition example $\frac{a+1}{x} + \frac{b-1}{x}$. Do these two rational expressions each have the same polynomial for a denominator? Add the numerators. Is the sum of the numerators $a + b$? Then is $\frac{a+b}{x}$ the sum of the two given rational expressions?

3. Look at the example below. When the definition of addition is applied, the numerator of the sum is $(3x - 4) + (5x + 3)$, or $8x - 1$, and the denominator is $2x + 1$.

$$\frac{3x-4}{2x+1} + \frac{5x+3}{2x+1} = \frac{(3x-4)+(5x+3)}{2x+1} = \frac{8x-1}{2x+1}$$

Test the result by replacing x by 1; by 3. What replacement for the variable is not permitted?

EXERCISES

Find the sums. Reduce them, if possible.

	a	b
1.	$\frac{x}{3} + \frac{y}{3}$	$\frac{2a}{5} + \frac{b}{5}$
2.	$\frac{3}{a} + \frac{5}{a}$	$\frac{4}{2b} + \frac{8}{2b}$
3.	$\frac{a}{a+b} + \frac{b}{a+b}$	$\frac{m}{m+8} + \frac{8}{m+8}$
4.	$\frac{3c}{5} + \frac{2c}{5}$	$\frac{x}{x+y} + \frac{y}{x+y}$
5.	$\frac{x^2}{x+1} + \frac{2x+1}{x+1}$	$\frac{x}{2(a+b)} + \frac{x+2}{2(a+b)}$
6.	$\frac{a^2}{a-b} + \frac{-b^2}{a-b}$	$\frac{3a}{y} + \frac{-3a}{y}$
7.	$\frac{2y}{3x} + \frac{2x}{3x} + \frac{3x}{3x}$	$\frac{2-x}{a+b} + \frac{x-2}{a+b}$
8.	$\frac{3p}{5-p} + \frac{4p-1}{5-p}$	$\frac{y+3}{3y-3} + \frac{2y+6}{3y-3}$

Subtraction

1. Check to see whether $\frac{x-2}{a} + \frac{-x+2}{a} = 0$ is a true statement. Are $\frac{x-2}{a}$ and $\frac{-x+2}{a}$ additive inverses?

Two rational expressions are additive inverses if they have a common denominator and their numerators are polynomials that are additive inverses.

2. Give the additive inverses of the following expressions.

a. $\frac{a}{b}$ **b.** $\frac{3-x}{y}$ **c.** $\frac{x^2-2x+3}{x+5}$ **d.** $\frac{2x^2+3x-7}{-3x+7}$

3. You will recall that the operation of subtraction (the inverse of addition) is defined as the addition of the additive inverse of the subtrahend to the minuend. Look at the example $\frac{x+a}{b} - \frac{a+1}{b}$. What is the additive inverse of $\frac{a+1}{b}$? What is the sum of $\frac{x+a}{b}$ and $\frac{-a-1}{b}$? Is it true that $\frac{x+a}{b} - \frac{a+1}{b} = \frac{x-1}{b}$?

EXERCISES

Perform the indicated operations. Reduce, if possible.

	a	b	c
1.	$\frac{x+2}{x-2} - \frac{2}{x-2}$	$\frac{2a}{b} + \frac{a-2}{b}$	$\frac{r}{r+5} - \frac{r}{r+5}$
2.	$\frac{2m}{m+n} + \frac{2n}{m+n}$	$\frac{z^2}{z-3} - \frac{9}{z-3}$	$\frac{3y-2}{y^2+1} + \frac{2}{y^2+1}$
3.	$\frac{x^2+y^2}{x+y} + \frac{2xy}{x+y}$	$\frac{a^2}{a+b} - \frac{b^2}{a+b}$	$\frac{y^2+1}{y-1} + \frac{-2y}{y-1}$
4.	$\frac{3a}{b} + \frac{2a}{b} - \frac{a}{b}$	$\frac{a+b}{b} - \frac{a-b}{b}$	$\frac{x^2}{x^2+y^2} - \frac{y^2}{x^2+y^2}$

5. a. $\frac{2x}{x^2-3x+2} - \frac{x-2}{x^2-3x+2}$ **b.** $\frac{2x-1}{x^2-x-6} - \frac{x-3}{x^2-x-6}$

The Least Common Denominator

1. What are the factors of $2ab$? of $3bc$? Then what factors must the least common multiple of the two expressions have? Is $6abc$ the least common multiple of $2ab$ and $3bc$?

▶ The least common denominator (L.C.D.) of two rational expressions is the least common multiple (L.C.M.) of the two denominators.

2. What is the L.C.D. of $\frac{3}{2ab}$ and $\frac{5}{3bc}$? If we now desire to express each of these rational expressions as equivalent expressions that have a common denominator, we make use of the identity element for multiplication. Is $\frac{3c}{3c}$ equal to 1? Then can $\frac{3}{2ab}$ be expressed as $\frac{3c}{3c} \cdot \frac{3}{2ab}$? Replace each question mark with an expression that will make a true sentence.

a. $\frac{3c}{3c} \cdot \frac{3}{2ab} = ?$ **b.** $\frac{?}{?} \cdot \frac{5}{3bc} = \frac{10a}{6abc}$

Mixed Expressions

1. You know that a number such as $4\frac{2}{3}$ is a mixed number. Can $4\frac{2}{3}$ mean the sum of the integer 4 and the rational number $\frac{2}{3}$? Can $4\frac{2}{3}$ also be expressed as the quotient of two integers? Does $4\frac{2}{3} = \frac{14}{3}$?

2. You have seen that polynomials are comparable to integers and that rational expressions are comparable to rational numbers. The indicated sum of a polynomial and a rational expression that is not a polynomial is called a **mixed expression.** Do you see that a mixed expression is comparable to a mixed number?

3. When you add 4 and $\frac{2}{3}$, do you use the addition rule for rational numbers by thinking of 4 as $\frac{12}{3}$?

Is the indicated sum $x + \frac{1}{x}$ a mixed expression? Which term is a polynomial? Which is a rational expression that is not a polynomial? Can you express x as the quotient of two polynomials? Then $x + \frac{1}{x}$ can be expressed as $\frac{x}{1} + \frac{1}{x}$. What is the L.C.D.? Completing the addition, we have $\frac{x^2}{x} + \frac{1}{x} = \frac{x^2+1}{x}$. Do you see that a mixed expression can be treated as a special case of adding two rational expressions?

EXERCISES

Express each of the following as a rational expression.

	a	b	c
1.	$a - \frac{1}{a}$	$\frac{1}{a} + b$	$\frac{c}{b} + d + \frac{b}{c}$
2.	$x + \frac{1}{x+1}$	$x + \frac{1}{x-1}$	$\frac{x}{x+1} - 1$
3.	$1 + \frac{x-y}{x+y}$	$\frac{2}{x-1} + 2$	$\frac{x}{y} - 2 + \frac{y}{x}$
4.	$x + 1 + \frac{1}{x+1}$	$x - 1 - \frac{1}{x-1}$	$\frac{a+b}{a-b} + 1$

Use the distributive principle to find the products.

5. $\frac{1}{x}(x+y)$
6. $\frac{2}{m}(m^2 - m)$
7. $(x+1)\left(\frac{1}{x+1} + 2\right)$
8. $x^2\left(\frac{1}{x^3} - \frac{1}{x^2}\right)$
9. $(x+2)(x+1)\left(\frac{1}{x+2} + \frac{1}{x+1}\right)$
10. $x\left(\frac{1}{x} + \frac{1}{y}\right)$
11. $(x-3)(x+1)\left(\frac{1}{x-3} - \frac{1}{x+1}\right)$
12. $y^3\left(\frac{2}{y^3} - \frac{1}{y^2} - 3\right)$

Addition, Using the L.C.D.

1. In order to add $\frac{5}{6}$ and $\frac{4}{9}$, you have learned to first express them in common denominator form.

$$\frac{5}{6} = \frac{?}{18} \qquad \frac{4}{9} = \frac{?}{18} \qquad \frac{15}{18} + \frac{8}{18} = \frac{?}{18}$$

Now that you can find the least common denominator of rational expressions, does it seem reasonable to do so in order to add two expressions with different denominators? Let us consider the example $\frac{3}{2ab} + \frac{5}{3bc}$.

(1) $\frac{3}{2ab} + \frac{5}{3bc} = \frac{3c}{3c} \cdot \frac{3}{2ab} + \frac{2a}{2a} \cdot \frac{5}{3bc}$

(2) $= \frac{9c}{6abc} + \frac{10a}{6abc}$

(3) $= \frac{9c + 10a}{6abc}$

What multiplication element was used in step (1)? What principle was used in step (3)?

2. Let us find the sum of $\frac{1}{4a^2bc}$ and $\frac{5}{18ab^2c}$. First find the L.C.D. To do this, factor each denominator and then find the L.C.M. of all the factors in each. Do you arrive at $36a^2b^2c$ as the L.C.D., or L.C.M.? Express $\frac{1}{4a^2bc}$ as $\frac{?}{36a^2b^2c}$. Express $\frac{5}{18ab^2c}$ as $\frac{?}{36a^2b^2c}$. The addition example is now:

$$\frac{9b}{36a^2b^2c} + \frac{10a}{36a^2b^2c} = \frac{9b + 10a}{36a^2b^2c}$$

3. To add the following:

$$\frac{x}{x^2 - 9} + \frac{2}{x^2 + 2x - 3} = ?$$

(1) Factor the denominators:

$$\frac{x}{(x+3)(x-3)} + \frac{2}{(x+3)(x-1)}$$

(2) Find the least common denominator:

$$(x+3)(x-3)(x-1)$$

(3) Express each addend with the L.C.D. as a denominator:

$$\frac{x-1}{x-1} \cdot \frac{x}{(x+3)(x-3)} + \frac{x-3}{x-3} \cdot \frac{2}{(x+3)(x-1)}$$

(4) Combine the numerators:

$$\frac{x^2 - x}{(x+3)(x-3)(x-1)} + \frac{2x - 6}{(x+3)(x-3)(x-1)} = \frac{x^2 + x - 6}{(x+3)(x-3)(x-1)}$$

(5) Factor and reduce the sum:

$$\frac{x^2 + x - 6}{(x+3)(x-3)(x-1)} = \frac{(x-2)\cancel{(x+3)}}{\cancel{(x+3)}(x-3)(x-1)}, \text{ or } \frac{x-2}{(x-3)(x-1)}$$

(1)

EXERCISES

Find the sums. If possible, reduce.

1. $\frac{1}{ab} + \frac{2}{bc} + \frac{3}{ac}$
2. $\frac{a}{b} + \frac{b}{d} + \frac{1}{2}$
3. $\frac{1}{2b} + \frac{b-c}{b}$
4. $\frac{1+x}{xy^2} + \frac{1-y}{x^2y}$
5. $\frac{3b}{14a} + \frac{5b}{21a}$
6. $\frac{x}{zy} + \frac{y}{zx} + \frac{3}{xy}$
7. $\frac{1}{a+4} + \frac{1}{a-4}$
8. $\frac{a}{a+b} + \frac{b}{a-b}$
9. $\frac{1}{x^2-4} + \frac{1}{x+2}$
10. $\frac{1}{x^2-4} + \frac{1}{x-2}$
11. $\frac{4x}{x+4} + \frac{3x}{x+3}$
12. $\frac{2m}{m-2} + \frac{3m}{m-3}$
13. $\frac{2b}{b^2-4} + \frac{3}{b+2}$
14. $\frac{5x}{x^2-6x+9} + \frac{1}{x^2-x-6}$

15. After you have had some experience in adding rational expressions, you can combine some of the steps to shorten the amount of pencil work, but you should be aware at all times of the exact procedure you are using. Your work might look like the following. Follow the work closely and explain the procedures taken in each step.

(1) $\dfrac{3}{x^2-25}+\dfrac{2x}{x+5}+\dfrac{3x}{x-5}$

$$=\frac{3}{(x-5)(x+5)}+\frac{2x(x-5)}{(x-5)(x+5)}+\frac{3x(x+5)}{(x-5)(x+5)}$$

(2) $$=\frac{3+2x^2-10x+3x^2+15x}{(x-5)(x+5)}$$

(3) $$=\frac{5x^2+5x+3}{x^2-25}$$

Is the sum in step (3) in lowest terms?

Add.

16. $\dfrac{3}{x+1}+\dfrac{x+2}{x^2-1}=?$

17. $\dfrac{a-c}{ac}+\dfrac{c-x}{cx}+\dfrac{x-a}{ax}=?$

18. $\dfrac{1}{x^2-25}+\dfrac{x}{2x+10}+\dfrac{3x}{3x-15}=?$

19. $\dfrac{3}{2b^2+5b-12}+\dfrac{2}{4b^2-9}+\dfrac{4}{2b^2+11b+12}=?$

20. $\dfrac{2}{x}+\dfrac{5}{x^2}+\dfrac{3}{x^3}=?$

Subtraction, Using the L.C.D.

1. You learned on page 170 how to form the additive inverse of a rational expression. Write the additive inverse of each of the following.

a. $\dfrac{3x}{5x-2}$ **b.** $\dfrac{x-1}{x^2+4}$ **c.** $\dfrac{x^2-x+2}{x^2+x-2}$

2. You know that to subtract is to add the additive inverse of the subtrahend. Study this subtraction example.

(1) $\dfrac{3a}{a^2-9}-\dfrac{2a}{a+3}=\dfrac{3a}{(a+3)(a-3)}+\dfrac{-2a}{a+3}$

(2) $$=\frac{3a-2a(a-3)}{(a+3)(a-3)}$$

(3) $$=\frac{3a-2a^2+6a}{(a+3)(a-3)}$$

(4) $$=\frac{-2a^2+9a}{a^2-9}$$

Do you see that we changed the subtraction example to an addition example in step (1)? How? What is the L.C.D.? State the principles that permit the other steps.

EXERCISES

Complete the indicated operations.

1. $\dfrac{1}{x^2-4}-\dfrac{1}{x+2}$ **2.** $\dfrac{a}{a+b}-\dfrac{b}{a-b}$

3. $\dfrac{x}{x^2-25}-\dfrac{1}{3x+15}$ **4.** $\dfrac{2x-5}{x-2}+\dfrac{x}{4-2x}$

5. $\dfrac{1}{x^2-4}-\dfrac{1}{x-2}$ **6.** $\dfrac{3x}{x-2}+\dfrac{4x}{x+2}$

7. $\dfrac{p}{p^2-q^2}-\dfrac{1}{p-q}$ **8.** $\dfrac{9}{x^2-81}-\dfrac{1}{9-x}$

9. $\dfrac{5}{3a+4}-\dfrac{2}{4a-2}$ **10.** $\dfrac{6m-13}{m^2-5m+6}-\dfrac{5}{m-3}$

11. Is the set of rational expressions closed under addition? Give an example. Is it commutative? associative? Illustrate each principle. Is there an identity element for addition? What is it? Does every rational expression have an additive inverse? Then can you say that the set of rational expressions forms a commutative group under the operation of addition?

Another Method of Division

1. You know that a rational expression can be the indicated quotient of two polynomials. For example, can $\frac{x^2+3x+2}{x+1}$ be expressed as $(x^2+3x+2) \div (x+1)$? One method of completing the division is shown below.

$$\frac{x^2+3x+2}{x+1} = \frac{\overset{1}{\cancel{(x+1)}(x+2)}}{\underset{1}{\cancel{x+1}}}, \text{ or } x+2$$

2. Another method for performing a division example such as $x^2+3x+2 \div x+1$ is similar to the method of long division used to divide rational numbers. Study the model example below.

The first step was to divide the first term of the dividend (x^2) by the first term of the divisor (x). The result is the first term of the quotient (x).

Next, we multiplied the divisor by this term and subtracted the product from the dividend. What was the remainder? Where does the "2" in the expression $2x+2$ come from? Then we divided the first term of $2x+2$ by the first term of the divisor. The result is the second term of the quotient. Multiply the divisor by this term and subtract. What is the remainder? Is the division complete?

$$\begin{array}{r} x+2 \\ x+1\overline{)x^2+3x+2} \\ \underline{x^2+x} \\ 2x+2 \\ \underline{2x+2} \\ 0 \end{array}$$

3. Suppose we attempt to do the above example in the form shown at the right.

$$x+1\overline{)3x+x^2+2}$$

Would this method of long division be possible? Try it. Do you see the importance of having both the divisor and dividend in the standard polynomial form?

Suppose the example you wish to work is $x^3-1 \div x-1$. Let us complete the division, but we will write the example as shown at the right.

Do you see that it helps to remember the terms that have 0 coefficients? It is not necessary, however, to write such terms. Merely leaving space where they would occur is sufficient.

$$\begin{array}{r} x^2+x+1 \\ x-1\overline{)x^3+0\cdot x^2+0\cdot x-1} \\ \underline{x^3-x^2} \\ x^2 \\ \underline{x^2-x} \\ x-1 \\ \underline{x-1} \\ 0 \end{array}$$

4. To perform the division $2x^2+3x+1 \div x+2$, we have the calculations shown below.

$$\begin{array}{r} 2x-1 \\ x+2\overline{)2x^2+3x+1} \\ \underline{2x^2+4x} \\ -x+1 \\ \underline{-x-2} \\ 3 \end{array}$$

How do you find "$2x$" in the quotient?
How do you find "-1" in the quotient?
How do you get "$-x-2$"?
What is the remainder?

EXERCISES

Divide.

1. $\frac{x^2+5x+6}{x+2}$
2. $\frac{x^2-5x+1}{x+1}$
3. $\frac{y^2+2y-3}{y-3}$
4. $\frac{a^2-7a-1}{a+1}$
5. $\frac{1+a^2-5a}{a-2}$
6. $\frac{r^2-5r-6}{r-4}$
7. $\frac{m^2+5m-8}{m+3}$
8. $\frac{x^2-3xy-4x^2}{x+y}$
9. $\frac{x^2+5xy-2y^2}{x-y}$
10. $\frac{2x^2-2x+3}{x+1}$
11. $\frac{-3+y+3y^2}{y-1}$
12. $\frac{4x^2-4x-3}{2x+1}$
13. $\frac{4x^2-4x}{2x+1}$
14. $\frac{x^2+1}{x+1}$
15. $\frac{x^3-6x^2+11x}{x-1}$

Complex Rational Expressions

1. You know that the quotient of two rational numbers is also a rational number; for example, $\frac{3}{4} \div \frac{5}{7} = \frac{3}{4} \cdot \frac{7}{5}$, or $\frac{21}{20}$. You also know that you may express this division example as $\frac{\frac{3}{4}}{\frac{5}{7}}$. Since at least one of the terms of this fraction is a fraction, this form is a *complex fraction*. Similarly, the quotient of two rational expressions is called a **complex rational expression.**

List the expressions in **a–d** that are complex rational expressions.

a. $\frac{\frac{4x^2}{5}}{7}$ **b.** $\frac{9abc}{11c^2}$ **c.** $\frac{\frac{1}{2}x}{7y}$ **d.** $\frac{\frac{rs^2}{s}}{\frac{st}{r}}$

2. Look at the following division example: $\dfrac{3ab}{2x^2y} \div \dfrac{12a^2b}{8xy^3}$. Written as a complex rational expression, it is $\dfrac{\dfrac{3ab}{2x^2y}}{\dfrac{12a^2b}{8xy^3}}$. Notice that one fraction bar is heavier and longer than the others. Why should it be this way?

The complex rational expression $\dfrac{\dfrac{3ab}{2x^2y}}{\dfrac{12a^2b}{8xy^3}}$ can be reduced (the indicated division performed) as illustrated at the right. How are $\dfrac{12a^2b}{8xy^3}$ and $\dfrac{8xy^3}{12a^2b}$ related?

$$\frac{\dfrac{3ab}{2x^2y}}{\dfrac{12a^2b}{8xy^3}} = \frac{3ab}{2x^2y} \div \frac{12a^2b}{8xy^3}$$

$$= \frac{3ab}{2x^2y} \cdot \frac{8xy^3}{12a^2b}$$

$$= \frac{\overset{1}{\cancel{3ab}} \cdot \overset{1}{2} \cdot \overset{y^2}{\cancel{4xy^3}}}{\underset{1}{\cancel{2x^2y}} \cdot \underset{1}{3} \cdot \underset{1}{\cancel{4a^2b}}}, \text{ or } \frac{y^2}{ax}$$

3. Study this example of the reduction of a complex rational expression. Explain each step.

$$\frac{\dfrac{3x^2}{x^2-1}}{\dfrac{x}{x+1}} = \frac{3x^2}{x^2-1} \cdot \frac{x+1}{x}$$

$$= \frac{3 \cdot \overset{x}{\cancel{x^2}} \cdot \overset{1}{\cancel{(x+1)}}}{\underset{1}{\cancel{(x+1)}}(x-1) \cdot \underset{1}{\cancel{x}}}, \text{ or } \frac{3x}{x-1}$$

4. There is another method of reducing a complex rational expression. Let us repeat the example in **3**, using this second method.

$$\frac{\dfrac{3x^2}{x^2-1}}{\dfrac{x}{x+1}} = \frac{\dfrac{3x^2}{(x-1)(x+1)}}{\dfrac{x}{x+1}} \cdot \frac{(x-1)(x+1)}{(x-1)(x+1)}$$

$$= \frac{\overset{x}{\cancel{3x^2}}}{\underset{1}{\cancel{x}}(x-1)}, \text{ or } \frac{3x}{x-1}$$

Why was each term multiplied by $(x-1)(x+1)$? This method of reducing a rational expression is particularly useful if either (or both) of the terms is a mixed expression.

EXERCISES

Reduce each of the following to a single rational expression. Use the method that seems easier for each example.

1. $\dfrac{\dfrac{3p^2}{4q}}{\dfrac{9p}{16q^2}}$

2. $\dfrac{\dfrac{33r^2s}{7ab}}{\dfrac{11rs^2}{28a^2b}}$

3. $\dfrac{\dfrac{x+y}{x}}{\dfrac{x-y}{x}}$

4. $\dfrac{\dfrac{y+2}{2y}}{\dfrac{y-2}{y}}$

5. $\dfrac{\dfrac{x^2-16}{x^2}}{\dfrac{x-4}{x}}$

6. $\dfrac{\dfrac{y}{16-y^2}}{\dfrac{y^2}{4+y}}$

7. $\dfrac{\dfrac{z}{z^2-1}}{\dfrac{2z^2}{z-1}}$

8. $\dfrac{\dfrac{x+y}{x-y}}{\dfrac{x-y}{x+y}}$

9. $\dfrac{\dfrac{3a+3b}{a^2-b^2}}{\dfrac{3}{a-b}}$

10. $\dfrac{\dfrac{1}{x}-1}{\dfrac{1}{x}+1}$

11. $\dfrac{1-\dfrac{1}{x^2}}{1+\dfrac{1}{x}}$

12. $\dfrac{\dfrac{x}{2}-\dfrac{2}{x}}{\dfrac{1}{2}+\dfrac{1}{x}}$

13. $\dfrac{\dfrac{y}{3}-\dfrac{3}{y}}{\dfrac{1}{3}-\dfrac{1}{y}}$

14. $\dfrac{\dfrac{1}{a+1}+1}{\dfrac{1}{a+1}-1}$

15. $\dfrac{\dfrac{x^2+1}{x}+2}{x-\dfrac{1}{x}}$

16. $\dfrac{p^2-\dfrac{1}{2p+1}}{p+\dfrac{1}{p^2-1}}$

USING RATIONAL EXPRESSIONS

Equations

Some equations contain rational expressions. Let us use our knowledge of rational expressions in solving such equations.

1. Consider the equation $\frac{5}{6} - \frac{1}{x} = \frac{2}{3x}$. What restrictions must be applied to the replacement set for x? You know that we can solve this equation if we can find an equivalent equation that we know how to solve by the methods learned in Chapter 3. Will multiplying each member of the equation by the L.C.D. give us the desired equation? What is the L.C.D. of 6, x, and $3x$? Now we can solve the equation as follows:

$$\frac{5}{6} - \frac{1}{x} = \frac{2}{3x}$$
$$6x \cdot \left(\frac{5}{6} - \frac{1}{x}\right) = 6x \cdot \left(\frac{2}{3x}\right)$$
$$5x - 6 = 4$$
$$5x = 10$$
$$x = 2$$

What is the solution? the solution set? Does the solution make the original equation true?

Was each term in the left member multiplied by $6x$? What principle was applied? Was the right member multiplied by $6x$? In solving equations of this type, be sure that each term in the equation is multiplied by the L.C.D.

Solve the following equations and check your solutions.

2. $\frac{3}{x} = 7$

3. $\frac{1}{2} + \frac{1}{3} = \frac{1}{x}$

4. $\frac{1}{2} - \frac{1}{x} = \frac{1}{3}$

5. $\frac{1}{2x} = \frac{1}{3x} + 1$

6. $\frac{2}{x} - 1 = 1$

7. $\frac{2}{x} + 1 = \frac{-1}{x}$

8. $\frac{-10}{t} - \frac{8}{t} = 1$

9. $\frac{5}{2x} + \frac{1}{12} = \frac{3}{x}$

10. $D + \frac{1}{D} = \frac{D^2 - 1}{D} + 1$

11. $\frac{1}{x} = \frac{2}{x} - 3$

12. Let us try to solve a more difficult equation involving rational expressions. $\frac{1}{y-4} - \frac{1}{y+2} = \frac{2}{y^2-4}$

What is the L.C.D.? What are the restrictions on the value of y? Then we have:

$$(y-4)(y+2)(y-2)\cdot\left(\frac{1}{y-4} - \frac{1}{y+2}\right) = (y-4)(y+2)(y-2)\cdot\frac{2}{(y+2)(y-2)}$$
$$(y+2)(y-2) - (y-4)(y-2) = (y-4)2$$
$$y^2 - 4 - y^2 + 6y - 8 = 2y - 8$$
$$6y - 12 = 2y - 8$$
$$4y = 4$$
$$y = 1$$

Test the solution in the original equation. What is the solution set of the original equation? Is the equation true when y is replaced by 1? Explain each step in the solution.

EXERCISES

Solve the following equations and check the solutions. Give the restrictions on the replacement sets for the variables.

1. $\frac{3}{x} - \frac{2}{5x} = 13$

2. $\frac{3m-5}{4} - \frac{2}{3m} = \frac{9m-7}{12}$

3. $\frac{2}{5x-1} = \frac{2}{3x+5}$

4. $\frac{1}{y+1} + \frac{1}{y-1} = \frac{4}{y^2-1}$

5. $\frac{2x-3}{4x-5} = \frac{4}{7}$

6. $\frac{x}{x-3} = \frac{x+1}{x-5}$

7. $x + \frac{1}{x} = \frac{3+x+x^2}{x}$

8. $\frac{1}{x+1} - \frac{1}{1-x} = \frac{1}{x^2-1}$

9. $\frac{x}{x+1} - \frac{1}{x} = 1$

10. $\frac{x}{x+1} = \frac{x+1}{x} + \frac{5}{x}$

11. $\frac{10+2t}{5t} = \frac{2}{t} - \frac{2}{5}$

12. $\frac{5x+8}{x^2-4} + \frac{24}{2-x} + \frac{45}{x+2} = 0$

13. Find a number such that 5 times its reciprocal diminished by twice its reciprocal equals 1 more than its reciprocal.

14. It is not possible to have two consecutive integers such that the sum of their reciprocals is equal to the reciprocal of their product. Show why this is not possible.

Transforming Formulas

1. You know that the formula for the area enclosed by a trapezoid is $A = \frac{1}{2}h(B + b)$ where A is the area, h the height, B the length of the longer base, and b the length of the shorter base. If you know the values of h, B, and b, it is easy to find the area. For example, suppose $h = 7$ inches, $B = 11$ inches, and $b = 9$ inches. What is the area enclosed by the trapezoid?

On the other hand, suppose you know that $A = 70$ square inches, $B = 11$ inches, and $b = 9$ inches. Let us see how you might find the height by expressing h in terms of A, B, and b. To do this, we **transform** the formula; in this case we express the formula in a form that has h as the **subject.** Follow the steps below that transform the formula.

$$A = \frac{1}{2}h(B + b)$$

$$A \cdot \frac{1}{B+b} = \frac{1}{2}h(B + b) \cdot \frac{1}{B+b}$$

$$\frac{A}{B+b} = \frac{1}{2}h$$

$$\frac{A}{B+b} \cdot 2 = \frac{1}{2}h \cdot 2$$

$$h = \frac{2A}{B+b}$$

Describe the operations performed in each step. Use the transformed formula to calculate the height of the trapezoid.

In your future work you will often find that by transforming formulas you can simplify computations.

2. Suppose you are given the formula $\frac{1}{D} + \frac{1}{d} = \frac{2}{R}$ and asked to transform it to make R the subject. Explain the operation in each step in this transformation.

$$dR + DR = 2Dd$$

$$R(d + D) = 2Dd$$

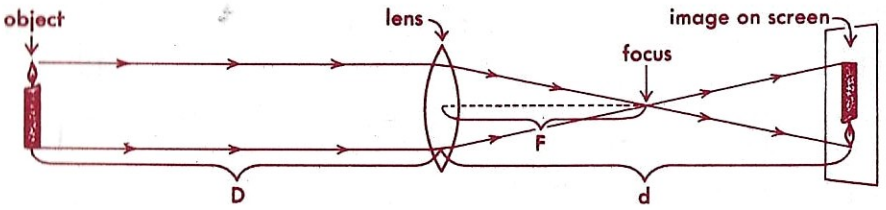
$$R = \frac{2Dd}{d+D}$$

Now transform $\frac{1}{D} + \frac{1}{d} = \frac{2}{R}$ to express d as the subject.

3. $S = \frac{n(a+l)}{2}$ is a formula used in algebra.
   a. Transform this formula to express n as the subject.
   b. Transform it to express a as the subject.
   c. Transform it to express l as the subject.

4. $C = \frac{en}{R+nr}$ is a formula used in electrical measurement. Express the formula so that the subject is R; r; n; e.

Lens Problems



The diagram above shows how a convex lens forms an inverted image of a candle. You can see that the light waves from the candle pass through the lens and converge at the focus. The distance, indicated by F, from the lens to the focus is known as the focal length.

Scientists have found that the distance of the object from the lens (D), the distance of the image from the lens (d), and the focal length (F) are related as shown by this equation. $\frac{1}{D} + \frac{1}{d} = \frac{1}{F}$

1. Find D if $d = 5$ inches and $F = 3$ inches.
2. Find d if $D = 2$ feet and $F = 1\frac{1}{3}$ feet.
3. Find F if $D = 18$ inches and $d = 9$ inches.
4. Prove that F cannot be equal to d.
5. Prove that F cannot be equal to D.
6. Suppose that the object is 10 times as far from the lens as its image and the focal length is 5 inches. Find the object and image distances from the lens.
7. Show that, if D and d are each doubled, F must also be doubled.

8. Use *increases, decreases,* or *remains the same* in the spaces below so that each sentence is true.

a. If D remains the same and d is increased, then F ___?___. You can get the answer by trying two sets of values of D and d and seeing what happens to F, or you can think: If d increases, $\frac{1}{d}$ ___?___; $\frac{1}{D} + \frac{1}{d}$ ___?___; $\frac{1}{F}$ ___?___; and F ___?___.

b. If D decreases and d remains the same, F ___?___.

c. If F increases and D remains the same, d ___?___.

d. If D and d each decrease, F ___?___.

Number Problems

1. Tom has marks of 75, 80, and 85. He is determined to make nothing but 100's in the future. We wish to find how many successive 100's he must get to achieve an average of 90.

Let x represent the number of 100's he must get.

Then $100x$ would be the sum of these 100's.

The total of all marks is $100x +$ ___?___.

The number of tests is $x +$ ___?___.

Since the average should be 90, the equation is:

$$\frac{100x + 240}{x + 3} = 90$$

Complete the solution and check.

2. A family drove 250 miles on the first day of a trip and 400 miles on the second day. They planned thereafter to drive 275 miles each day. In how many days will their average be 300 miles per day for the entire trip?

3. A manufacturer divides a bar into two parts in such a way that one part is 12 inches longer than the other and also so that the ratio of the lengths is 5:3. How long was the bar?

4. Suppose that in **3** one part is 3 inches more than twice the other and the ratio of their lengths is 9:4. Find the length of the bar.

5. When one number D is divided by another number d, the formula that relates the numbers with their quotient q and the remainder R is $\frac{D}{d} = q + \frac{R}{d}$. For example, is it true that $\frac{15}{4} = 3 + \frac{3}{4}$?

Use this formula to solve the following problem: One number is 32 more than another. If the larger is divided by the smaller, the quotient is 2 and the remainder is 6. Find the two numbers.

6. One number is 4 less than 3 times a smaller number. If the larger is divided by the smaller, the quotient is 2 and the remainder is 9. Find the numbers.

Mixture Problems

1. A druggist has 100 ounces of a solution made by dissolving iodine in alcohol. In these 100 ounces, 5 ounces are iodine and 95 ounces are alcohol. This is referred to as a 5% iodine solution. Would the strength (the per cent) of this solution increase, or decrease, if:

a. more alcohol is added?

b. more iodine is added?

c. alcohol is evaporated?

The strength of a solution is found by dividing the measure of the ingredient under consideration by the measure of the total solution. These measures may be of weight or of volume. For example, if i is the measure of one ingredient and s is the measure of the solution, the strength of the solution is $\frac{i}{s}$, or $100\,\frac{i}{s}\,\%$.

2. What number represents i in **1**? What number represents s in **1**? What per cent solution would the druggist have if he added 20 ounces of alcohol? What per cent if he allowed 20 ounces of alcohol to evaporate?

3. The druggist wished to weaken 4 ounces of his iodine solution to a 3% solution. How many ounces of pure iodine are there in 4 ounces of his 5% solution? This is the number of ounces of pure iodine. Will this number change if pure alcohol is added? Will the number of ounces in the total solution change? To determine the number of ounces of alcohol he should add:

Let $x =$ the number of ounces of alcohol to be added.

Then $\frac{.05 \cdot 4}{4 + x} = \frac{3}{100}$. Solve, and check your answer.

4. How many ounces of alcohol should the druggist evaporate in order to make 4 ounces of his 5% solution become an 8% solution?

Work Problems

1. Ann can mow a lawn in 4 hours. Dan can do the same lawn in 2 hours. Suppose they work together, each using a mower. Do you think that it will take them 4 + 2, or 6, hours? $\frac{1}{2}(4+2)$, or 3, hours? Explain why you think both these answers must be wrong.

Will Ann cut $\frac{1}{4}$ of the lawn in 1 hour? What part of the lawn will Dan cut in 1 hour? Then what part of the lawn will they both cut in 1 hour? If together they can cut the lawn in x hours, what part of the lawn can they cut together in 1 hour? Do you see that $\frac{1}{4}+\frac{1}{2}=\frac{1}{x}$? We can organize our analysis by using the following table.

	Ann	Dan	Together
Hours needed:	4	2	x
Part mowed in 1 hour:	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{x}$

Then $\frac{1}{4}+\frac{1}{2}=\frac{1}{x}$

Solve this equation and check your solution.

2. A mason can build a wall in 10 hours. His helper can do it in 15 hours. How long will it take them together if each works at his own rate?

3. Joan can paint a fence in 5 hours. Tom can do it in 4 hours. If they work together at their own rates, how long will it take them to paint the fence?

Motion Problems

1. You know that when an object moves with a uniform motion of r miles an hour for t hours, it will cover a distance of rt miles. Expressed as a formula, this is $d = rt$. Express r in terms of d and t. Express t in terms of d and r.

2. A boy rides a bicycle a distance of 15 miles and returns on foot. His rate of riding a bicycle is 5 times his walking rate. The total time is 6 hours. Can we determine what his rate of walking is? his rate of riding?

Let x represent his rate of walking in miles per hour.
Then $5x$ represents his rate of riding in miles per hour.
Then the number of hours he walks is represented by $\frac{15}{x}$.

And the number of hours he rides is represented by $\frac{15}{5x}$.

Write an equation that says that the total time is 6 hours. Solve the equation and check your solution.

The analysis of the problem can be organized as follows:

	Rate	Distance	Time
Boy walking	x	15	$\frac{15}{x}$
Boy riding	$5x$	15	$\frac{15}{5x}$

Then $\frac{15}{x}+\frac{15}{5x}=6$

What is the boy's rate of walking? of riding?

3. A man drove a car a distance of 200 miles. He made the return trip in an airplane over the same route. The average rate of flying was 6 times that of driving. If the round trip was made in 5 hours, what was the average speed of flying?

4. Todd finds that he can row 9 miles downstream in the same time it takes him to row 3 miles upstream. He knows that he can row 4 miles per hour in still water. Let c equal the rate of the current in miles per hour. When he rows downstream, does the current add to his speed? Then does $4 + c$ represent Todd's rate of rowing downstream? When he rows upstream, does the current slow him down? Then how do we represent the rate when Todd rows upstream? Represent the time it takes to row 9 miles downstream; 3 miles upstream.

Write the equation that says that these times are equal. Solve and check. This analysis can be organized as follows:

	Rate	Distance	Time
Downstream	$4+c$	9	$\frac{9}{4+c}$
Upstream	$4-c$	3	$\frac{3}{4-c}$

Equation: $\frac{9}{4+c}=\frac{3}{4-c}$

5. Jerry can row 5 miles an hour in still water. He finds that he can row 4 miles downstream in the same time it takes him to row 1 mile upstream. Find the rate of the current.

6. Charles knows his motorboat can travel 10 miles per hour in still water. He wants to find the rate of the current of a river. He finds it takes him 3 times as long to travel 6 miles upstream as it takes him to return. Find the rate of the current.

EXERCISES

1. How much water should be added to 3 quarts of a disinfectant solution that contains 20% of an active ingredient so that the result will be a 15% solution?

2. Brine is used to preserve fish. It is a solution of salt in water. A certain barrel contains 100 pounds of a 15% solution. How many pounds of water should be evaporated to form a 20% solution?

3. A certain airplane can spray a given region in 8 hours. A second airplane takes 2 more hours. How long will it take to spray the region if both airplanes are used?

4. Mr. O'Brien can plow his field in 40 hours. He hired a helper, and together they did it in 25 hours. How long would it have taken the helper alone to plow the field?

5. What number added to 3 and to 5 produces two sums that have a ratio of 7:11?

6. A number is subtracted from the numerator and also from the denominator of $\frac{3}{2}$. The resulting fraction then has the value of $\frac{2}{3}$. Find the number.

7. A river steamer travels 80 miles downstream in the same time it takes the steamer to travel 55 miles upstream. The engine drives the steamer in still water at a rate that is 11 miles an hour more than the rate of the current. Find the rate of the current.

8. A dairyman has 300 pounds of milk testing 4.2% butterfat. How many pounds of skim milk (no butterfat) should he add to form an "approved milk" containing 3.2% butterfat?

9. A swimming pool is filled by two pipes. The first will fill it in 4 hours. Both pipes together fill it in $2\frac{1}{2}$ hours. How long will it take the second pipe alone to fill the pool?

10. A pipe can fill a tank in 2 hours. A drain can empty it in 3 hours. Suppose both pipe and drain are opened. How long will it take to fill the tank?

11. A druggist has 50 ounces of a 6% iodine solution. How many ounces of a 3% solution should he mix with the 50 ounces to make a 4% solution?

Do the 50 ounces of the 6% solution contain $.06 \times 50$ ounces of iodine? Do x ounces of the 3% solution contain $.03x$ ounces of iodine? Then how many ounces of iodine will the mixture contain? How many ounces of solution will there be altogether? Write the equation that says that the mixture will be a 4% solution. Solve and check.

12. How many pounds of milk testing 4% butterfat should a farmer mix with 100 pounds of milk testing 3.2% butterfat to obtain a mixture containing 3.5% butterfat?

13. Carl knows that in still water he can swim twice as fast as the rate of the current of the stream that passes his cottage. He swims $\frac{1}{2}$ mile upstream and back in a total time of 40 minutes. What is the rate of the current?

14. Joe can do a job in 6 hours. Peter can do it in 4 hours. If Joe works 4 hours, what part of the job has he done? If Peter works 1 hour, what part of the job has he done? Have they finished the job under these circumstances? What part of the job is yet to be done?

15. If Peter works x hours and Joe works 1 hour more $(x + 1)$, write the expressions that tell which part each of them does. Find the value of x which will result in the job being completed under these circumstances.

16. Suppose Joe works alone for 2 hours and leaves the job. Peter then finishes it alone in how many hours?

17. A plane can fly 200 miles per hour when there is no wind. It can travel 850 miles with the wind in the same time it can travel 500 miles against the same wind. Find the wind velocity.

18. A nurse has 10 ounces of a 20% Argyrol solution. How many ounces of a 30% solution should she add to produce a mixture of 28%?

19. A pipe can fill a tank in 3 hours, and its drain can empty it in 4 hours. The pipe is opened for 1 hour and then the drain is also opened. In how many hours will the tank be filled?

20. To 16 ounces of a 6% solution of iodine, x ounces of pure iodine are added. Then x ounces of alcohol are evaporated from the result. What is the value of x if the final solution is 9%?

NEGATIVE EXPONENTS AND SCIENTIFIC NOTATION

Negative Exponents

1. From your work earlier in the chapter, do you agree with the results of the example at the right?

$$\frac{x^4}{x^5} = \frac{x \cdot x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} = \frac{1}{x}$$

In a similar way, express the following.

a. $\frac{y^3}{y^4}$ **b.** $\frac{d^2}{d^3}$ **c.** $\frac{r^5}{r^6}$

Remember, each variable represents a non-zero number. Why?

Do you agree with the results shown below?

$$\frac{y^3}{y^4} = \frac{1}{y} \qquad \frac{d^2}{d^3} = \frac{1}{d} \qquad \frac{r^5}{r^6} = \frac{1}{r}$$

Perhaps it has occurred to you that the examples were of the form $\frac{a^m}{a^n} = a^{m-n}$. But $\frac{a^m}{a^n} = a^{m-n}$ has been defined only if $m \geq n$. Is this true in the above examples?

2. Suppose you had not realized that $\frac{a^m}{a^n}$ was defined only for $m \geq n$. Then in part **a** of **1** you might have decided to subtract exponents. You would have written: $\frac{y^3}{y^4} = y^{3-4}$. What does $3 - 4$ equal? Then your answer would be y^{-1}. But an exponent of "-1" has no meaning as *exponent* has been defined so far. Our exponents have all been members of $\{0, 1, 2, 3, \ldots\}$.

Suppose you had decided to subtract exponents in working with parts **b** and **c** in **1**. You would have written: $\frac{d^2}{d^3} = d^{2-3}$, or d^{-1}; $\frac{r^5}{r^6} = r^{5-6}$, or r^{-1}. Again in each case a "-1" would have been obtained as an "exponent."

3. Let us see whether we can establish a meaning for an exponent of "-1." In **1**, you found that $\frac{y^3}{y^4} = \frac{1}{y}$; $\frac{d^2}{d^3} = \frac{1}{d}$; and $\frac{r^5}{r^6} = \frac{1}{r}$. Then y^{-1}, if it is to be consistent with your earlier work, must equal $\frac{1}{y}$; d^{-1} must equal $\frac{1}{d}$; and r^{-1} must equal $\frac{1}{r}$.

4. Below are two examples. Do you agree with the results?

$$\frac{c^2}{c^4} = \frac{c \cdot c}{c \cdot c \cdot c \cdot c}, \text{ or } \frac{1}{c^2} \qquad \frac{e^3}{e^5} = \frac{e \cdot e \cdot e}{e \cdot e \cdot e \cdot e \cdot e}, \text{ or } \frac{1}{e^2}$$

5. Now suppose you were given the expressions $\frac{c^2}{c^4}$ and $\frac{e^3}{e^5}$ and remembered the form $\frac{a^m}{a^n} = a^{m-n}$, but not the restriction $m \geq n$.

Then you might have written $\frac{c^2}{c^4} = c^{2-4}$, or c^{-2}; $\frac{e^3}{e^5} = e^{3-5}$, or e^{-2}. Does an exponent of "-2" have any meaning thus far?

Now look back at the completed examples $\frac{c^2}{c^4} = \frac{1}{c^2}$; $\frac{e^3}{e^5} = \frac{1}{e^2}$.

Do you see that c^{-2}—if it is to have any meaning—must equal $\frac{1}{c^2}$; e^{-2} must equal $\frac{1}{e^2}$?

6. Using the expressions $\frac{x^2}{x^5}$, $\frac{y^3}{y^6}$, and $\frac{m^4}{m^7}$, find what meaning must be assigned to symbols such as "x^{-3}," "y^{-3}," and "m^{-3}."

Let us accept the agreement at the right for any positive integer m.

$$a^{-m} = \frac{1}{a^m} \quad (a \neq 0)$$

Then we have the following law of exponents.

For any non-zero number a and integers m and n, $\frac{a^m}{a^n} = a^{m-n}$.

7. Study this table of powers of ten. Is the pattern consistent; that is, is it true that for every n where n is an integer, $10^n = \frac{1}{10}$ of 10^{n+1}? Now let us test our agreement concerning negative exponents in other ways.

	Powers of ten
10^3	1000
10^2	100
10^1	10
10^0	1
10^{-1}	$\frac{1}{10}$, or .1
10^{-2}	$\frac{1}{10^2}$, or .01
10^{-3}	$\frac{1}{10^3}$, or .001

8. What does 2^3 equal? What does $\frac{1}{2^4}$ equal? What does $8 \cdot \frac{1}{16}$ equal? Now, using exponents: $2^3 \cdot 2^{-4} = 2^{3+(-4)} = 2^{-1}$, or $\frac{1}{2}$

Are the results of this method consistent with the results obtained by arithmetic?

9. Does $100 \div \frac{1}{1000}$ equal 10,000? Using exponents: $10^2 \div 10^{-3} = 10^{2-(-3)}$. Complete the work. Are the two results consistent?

EXERCISES

Simplify each of these expressions. Assume that the variable in each represents a non-zero number.

1. $y^2 \cdot y^{-2}$
2. $x^3 \cdot x^{-4}$
3. $r \cdot r^{-1}$
4. $\frac{x^2}{x^5}$
5. $\frac{y^2 \cdot y^{-1}}{y^3}$
6. $m^{-3} \cdot m^{-2}$
7. $\frac{d^{-2} \cdot d^{-1}}{d^3}$
8. $\frac{r^{-3}}{r^{-4}}$
9. $\frac{x^3}{x^{-4}}$
10. If any of your answers for Exercises **1–9** are expressed in terms of negative exponents, re-express them, using positive exponents.
11. Evaluate each of the following expressions at 4.
    a. x^{-3} b. $3 \cdot y^{-2}$ c. $(3y)^{-2}$
    d. $\frac{m^{-3}}{m^{-1}}$ e. $r \cdot r^{-3}$ f. $(x^{-5} \cdot x^6)(x^{-2})$
12. Express in decimal form: 10^{-4}; 10^{-5}; 10^{-6}; 10^{-7}.
13. Write each of the following as a power of ten: 1,000,000,000; 100,000,000; 10,000,000; 1,000,000; 100,000; 10,000.
14. Make a table of powers of ten. Enter the results of your work in Exercises **12** and **13**.

Scientific Notation

1. Are each of the following sentences true?
    a. $7000 = 7 \times 10^3$ b. $505 = 5.05 \times 10^2$
    c. $38 = 3.8 \times 10$ d. $.85 = 8.5 \times 10^{-1}$
    e. $.063 = 6.3 \times 10^{-2}$ f. $.008.1 = 8.1 \times 10^{-3}$
2. Express each of the following as the product of two factors—one a whole number or decimal mixed number less than ten and the other a power of ten.
    a. .65 b. .003 c. 7056
    d. .0007 e. .000083 f. .4
    g. 93,457 h. 3,000,000 i. .45632
3. In your reading, you have probably seen statements of this kind:

    The orbit of the earth is about 5.84×10^8 miles.
    The mass of a hydrogen atom is about 1.7×10^{-24} gram.

The numerals 5.84×10^8 and 1.7×10^{-24} are examples of **scientific notation.** This system is widely used in science and mathematics because it provides a convenient means of expressing very large and very small numbers.

To express a number in scientific notation, we express it as the product of two factors. One is a whole number or decimal mixed number less than ten. The other is a power of ten.

What is the conventional decimal numeral for 5.84×10^8? Try to express 1.7×10^{-24} as a conventional decimal numeral. Do you see that it is not convenient to write such a numeral or to perform computations with it?

EXERCISES

1. The earth is about 9.3×10^7 miles from the sun. Express this distance as a conventional numeral.
2. Light travels at a rate of about 1.86×10^5 miles per second. Express 1.86×10^5 as a conventional numeral.
3. A *light-year* is the name for the number of miles that light travels in a year. In scientific notation, a light-year is approximately 6×10^{12} miles. If light takes about 5 years to travel from a certain star to the earth, how many miles is it from this star to the earth? Express this distance in scientific notation. Remember, the first factor in your answer must be a whole number or decimal mixed number less than ten.
4. The approximate wave lengths of some different kinds of light are as follows: violet, .000039 cm; blue, .000046 cm; yellow, .000058 cm; orange, .000061 cm; red, .000077 cm. Express these lengths in scientific notation. Green light has a wave length of 5.4×10^{-5} cm. In the series of wave lengths of colors, the wave length of green is between the wave lengths of what two colors?
5. The diameter of a red blood cell is about 7.5×10^{-4} centimeter, and the diameter of a white blood cell is about 1.5×10^{-3} centimeter. Which diameter is greater? How many times as great? To find the answer to the last question, perform the indicated division: $\frac{1.5 \times 10^{-3}}{7.5 \times 10^{-4}}$. Can we write this as $\frac{1.5}{7.5} \times \frac{10^{-3}}{10^{-4}}$, or $\frac{1.5}{7.5} \times (10^{-3+4})$? Complete the work.

Perform the operations indicated. Reduce answers to lowest terms, if necessary. 158–165

1. $\frac{2x}{b} \cdot \frac{3a}{4x^2}$
2. $\frac{wx}{2t} \cdot \frac{6t^2}{3w}$
3. $\frac{2y-3}{3y^2+6y} \cdot \frac{y}{4y^2-9}$
4. $\frac{3-x}{6x+9} \cdot \frac{3+2x}{x^2-9}$
5. $\frac{x^2}{ac} + \frac{x}{ac}$
6. $\frac{2t^2}{x} + \frac{3t}{xy}$ 168–175
7. $\frac{a+b}{b-c} \cdot \frac{3a-b}{2b-c}$
8. $\frac{a}{y} - \frac{3y}{a}$
9. $\frac{2x}{5} - \frac{2}{x^2}$
10. $\frac{2x-1}{3a} - \frac{x+a}{3a}$
11. $\frac{x^2+2}{x^2+6x+9} + \frac{x-3}{3x+9}$
12. $\frac{2x}{x^2-1} - \frac{x}{x^2+5x+4}$
13. $y - \frac{2}{y-1}$
14. $9t + \frac{3t}{2-t}$
15. $3 + x - \frac{x}{x-1}$
16. $\frac{z^2-3z}{2z-1} \div \frac{z^2-9}{4z^2-1}$
17. $3x-4\overline{)6x^3-17x^2+9x+4}$ 166–167, 176–177
18. $\dfrac{\frac{2}{y}}{y-\frac{2}{y}}$
19. $\dfrac{\frac{2x+1}{x}}{\frac{4}{x^2}}$
20. $\dfrac{\frac{2y^2}{3}}{4y}$ 177–178

Solve the following equations. Check your answers. 180–181

21. $\frac{3x}{5} - \frac{x}{4} = \frac{7}{2}$
22. $\frac{5}{x} + \frac{3}{2x} = 2$
23. $\frac{y+3}{5} + \frac{2y-2}{6} = 6$
24. $\frac{1}{x+2} - \frac{1}{2-x} = \frac{3x+8}{x^2-4}$
25. $\frac{x}{x+1} + x = \frac{2x^2}{2+2x} + 4$
26. $\left(\frac{4}{x} - x\right)\frac{x}{x-2} = 3$

27. Express the formula $E = mc^2$ with c^2 as the subject. 182

28. Bob finds that on a stream flowing at 3 miles per hour it takes him as long to paddle 12 miles downstream as it takes to paddle 3 miles upstream. Find Bob's rate in still water. 187

29. Simplify the following expressions. 190–191

a. $y^{-3} \div y$ b. $a^2 \cdot \frac{1}{a} \cdot a^4$ c. $\frac{x^4}{x^6} \cdot \frac{x}{x^{-3}}$

30. Express the following in scientific notation. 192–193

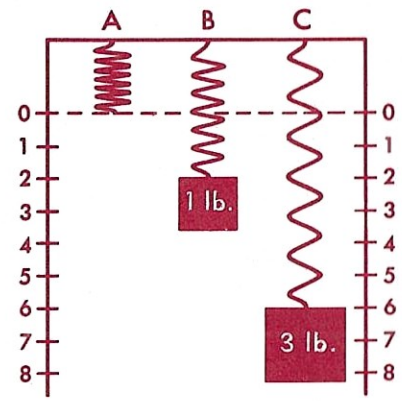
a. 4732 b. .15 c. 23,000,100 d. .0000683

If you had difficulty with any of the work on this test, find help on the pages listed at the ends of the rows.

6 Relations and Functions

Ordered Pairs of Numbers As you know, in performing experiments, a scientist collects data on his observations. Frequently, these data can be meaningfully organized in terms of pairs of numbers. For instance, consider a simple experiment you can perform yourself.

1. A spring is secured at position A. When there is no weight attached to it, the spring reaches a point that we let correspond to zero. When a weight is attached (positions B and C), the spring is stretched, and the distance it stretches can be measured using the indicated vertical scale. Suppose a 1-pound weight stretches the spring 2 inches, while a 3-pound weight stretches it 6 inches. We might record these two facts as follows: (1,2), and (3,6). In each pair of numbers, the first number indicates a number of pounds and the second number indicates a number of inches. What would (4,8) mean? $(\frac{1}{2},1)$?



2. As the experiment continues, more and more number pairs are collected. Suppose a weight of $1\frac{1}{2}$ pounds stretches the spring 3 inches. Would it be correct to record the pair as $(3,1\frac{1}{2})$? Why not? We should be careful to observe a correct order in writing such a pair of numbers.

We call a pair of numbers in which the order is meaningful an **ordered number pair.**

3. Study the number pairs we have so far.

$$(1,2),\ (3,6),\ (4,8),\ (\tfrac{1}{2},1),\ (1\tfrac{1}{2},3)$$

What pattern seems to be true for each pair of numbers? Let us agree that for each possible number pair in the experiment the number of inches (i) that the spring stretches is equal to twice the number of pounds (p) in the weight attached to the spring. Can we state the relationship mathematically as $i = 2p$?

4. Each of the following weights is attached to the spring. Tell the number of inches that the spring stretches in each case.

a. $2\frac{1}{2}$ pounds **b.** $3\frac{1}{2}$ pounds **c.** $4\frac{1}{2}$ pounds

Then the number pairs $(2\frac{1}{2},5)$, $(3\frac{1}{2},7)$, and $(4\frac{1}{2},9)$ are members of the set of results obtained in this experiment.

As you can see, the results in this experiment involve a set of ordered pairs. In mathematics, a set of ordered pairs is called a **relation.** An experimental scientist looks for a pattern, or formula, that describes how the numbers in each pair are related to each other. In mathematics, this pattern, or formula (for example, the sentence $i = 2p$), is called the **rule** of the relation.

5. For another illustration of a relation, look back at the work you did with "free fall" in Chapter 4 (pages 126–127). There you found that at the end of 1 second an object will have fallen 16 feet; at the end of 2 seconds, 64 feet; at the end of 3 seconds, 144 feet. If we agree to express the pairs of numbers as (number of seconds, number of feet), we have the ordered pairs (1,16), (2,64), and (3,144) as members of the set.

In the case of free fall, the pattern or rule of the relation is given by the sentence $s = 16t^2$. At the end of 5 seconds, how far will the object have fallen? at the end of 6 seconds? Write these data as number pairs using the agreement made above.

6. Is the sentence $i = 2p$ an open sentence with two variables? We speak of such a sentence as a sentence *in* two variables. Is $s = 16t^2$ a sentence in two variables? We have not spoken about the replacement sets for the variables in these sentences. In physical situations and scientific experiments, there are many factors that complicate the discussion of relations.

In the case of the spring, would it be reasonable to limit the replacement set for p to non-negative numbers? Why? Suppose, too, the spring cannot withstand a weight of more than 7 pounds. Then it would be meaningless to state that the number pair (8,16) is a member of the relation.

Let us continue our study of relations with simple mathematical sentences in two variables whose replacement sets are carefully described.

RELATIONS

Open Sentences and Relations

1. Suppose you are given the open sentence $y = x + 2$ and are told that x and y may be replaced by any member of the set {1, 2, 3, 4, 5}. When x is replaced by 1, which (if any) replacement for y will result in a true statement? When x is replaced by 2? by 3? by 4? by 5?

Let us agree that in any open sentence involving the variables x and y the x-replacements will be named first. Then is (1,3) a solution of the sentence $y = x + 2$? Is (2,4)? (3,5)? Why is (4,6) *not* a solution of the sentence? Is (0,2) a member of the solution set of the sentence? Why not?

Then the solution set of the sentence $y = x + 2$ is the set {(1,3), (2,4), (3,5)}. Do you see that we have a set of ordered pairs—that is, a relation? The sentence $y = x + 2$ is the rule of the relation. It determines a set of ordered pairs that belong to the relation.

2. Consider the same sentence, $y = x + 2$, but suppose the replacement set for x is {1, 2, 3, 4, 5} and the replacement set for y is {1, 2, 3, 4, 5, 6, 7}. Does the solution set consist of five ordered pairs instead of three? List these number pairs within braces. Do we now have a relation with five members?

Do you see that the solution set of an open sentence in two variables is a set of ordered pairs—that is, a relation?

3. Using the same sentence, $y = x + 2$, suppose the replacement set for both x and y is the set of rational numbers. Do you think there are more than five number pairs in the relation? more than ten number pairs? Name ten pairs that are members of the relation.

4. Do you agree that there are an unlimited number of pairs that satisfy $y = x + 2$ if the replacement set for x and y is the set of rational numbers?

You can see that we cannot make a list of all the members of the relation, so we denote the relation as follows:

$$\{(x,y) \mid y = x + 2\}$$

This is read "the set of all ordered pairs (x,y) such that $y = x + 2$." The rule $y = x + 2$ states the condition that must be satisfied for any number pair to be included in this relation.

5. Now consider the open sentence $y > x$, and let the replacement set for x be $\{0, 1, 2\}$ and the replacement set for y, $\{0, 1, 2, 3\}$. Is $(0,0)$ a member of the solution set of $y > x$? Why not? Is $(0,1)$ a member? $(0,2)$? $(0,3)$? Name all the other members of the solution set. How many members does the solution set have altogether?

You can see that here again we have a set of ordered pairs. Thus, an inequality such as $y > x$ may also serve as a rule for a relation. Is the following set the relation?

$$\{(0,1), (0,2), (0,3), (1,2), (1,3), (2,3)\}$$

The rule for a mathematical relation is often an open sentence. In that case, the solution set of the open sentence is the relation.

6. Suppose that for the sentence $y > x$ the replacement set for both x and y is the set of rational numbers. Does the solution set consist of an unlimited number of ordered pairs? As in **4**, we can indicate the relation whose rule is $y > x$ by simply writing $\{(x,y) \mid y > x\}$. How do you read this?

Of course, it must be clearly understood that the replacement set for the variables is as stated above.

EXERCISES

1. Suppose the replacement set for each variable is the set $\{1, 2, 3\}$. List all the number pairs in the relations whose rules are the following.

a. $x + y = 3$ **b.** $y = x$ **c.** $x = y + 1$
d. $y < x$ **e.** $y = x^2$ **f.** $x + y = 6$

2. Using set notation (as in **4** and **6**, page 198), state the relations whose rules are the following. The replacement set for each variable is the set of rational numbers. List three ordered pairs that are members of each relation.

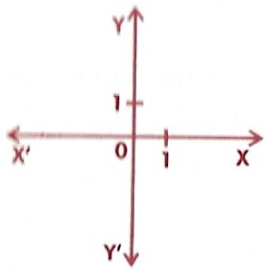
a. y is the square of x.
b. b is twice a.
c. p is 3 more than twice q.
d. The sum of x and y is 18.
e. The sum of three times x and one-half y is 22.
f. y is greater than twice x.

3. In each of the following exercises, the ordered pairs are members of a relation. Find an open sentence that can serve as the rule for each relation. Let x serve as the variable representing the first number in each pair and y the second number in each pair.

a. $(3,4)$, $(4,5)$, $(12,13)$, $(-1,0)$, $(-3,-2)$
b. $(0,0)$, $(1,3)$, $(12,36)$, $(-4,-12)$, $(\frac{1}{2},1\frac{1}{2})$
c. $(4,3)$, $(5,4)$, $(13,12)$, $(0,-1)$, $(-2,-3)$
d. $(0,0)$, $(3,1)$, $(36,12)$, $(-12,-4)$, $(1\frac{1}{2},\frac{1}{2})$
e. $(1,3)$, $(2,5)$, $(3,7)$, $(4,9)$, $(5,11)$

Graphs of Relations

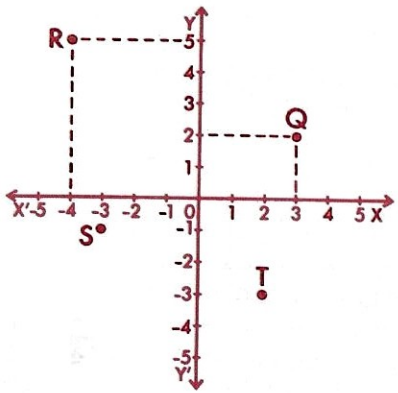
The study of mathematical relations can be simplified by graphs. As you know, a graph is a drawing that shows a relationship between certain sets of numbers. Bar graphs, broken-line graphs, circle graphs, and other kinds of graphs help to picture sets of business and economic data. Mathematical relations are pictured by a method that was first used systematically by René Descartes (1596–1650), a French mathematician and philosopher. This method leads to a branch of mathematics called analytic geometry and is useful in the study of calculus.



We start with two lines $X'X$ (read "X prime, X") and $Y'Y$, each perpendicular to the other. We call $X'X$ the **x-axis** and $Y'Y$ the **y-axis.** They intersect in a point named O, which is called the **origin.** It is customary to draw $Y'Y$ as a vertical line and $X'X$ as a horizontal line.

Our next step is to choose a unit length and to construct a number line on $X'X$ in the way that is familiar to you. We assign the number 0 to the origin, and the number 1 is assigned to the point one unit to the right of the origin. Is each point to the right of 0 then associated with a positive number? Is each point to the left of 0 associated with a negative number? Tell where on the x-axis you would locate the point associated with 9; with -1; with -3; with $-4\frac{1}{2}$.

On $Y'Y$ we also construct a number line, with the number 0 again assigned to the origin. By custom, we assign the number 1 to the point one unit *above* the origin. Then all positive numbers such as 2, 3, 4, and so on, are associated with points above the origin. It follows that on the y-axis negative numbers are associated with points *below* the origin. Tell where on the y-axis you would locate the point associated with $3\frac{1}{2}$; with -2; with $-3\frac{1}{2}$; with $-\frac{1}{2}$.

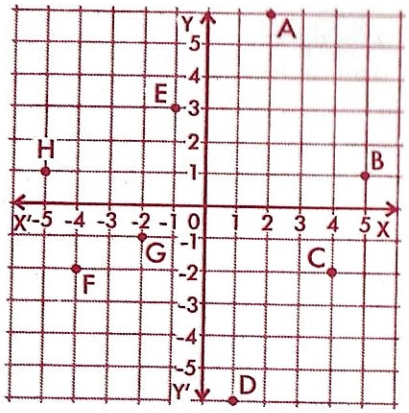


1. The lines $X'X$ and $Y'Y$ are called **coordinate axes**, and they determine a plane which is called the **coordinate plane.** Using the axes, we assign an ordered pair of numbers to a point Q as follows: From Q, perpendiculars are drawn to the x-axis and to the y-axis. What number corresponds to the point where the vertical perpendicular intersects the x-axis? This number, 3, is called the ***x*-coordinate**, or **abscissa**, of Q. What number corresponds to the point where the horizontal perpendicular intersects the y-axis? This number, 2, is called the ***y*-coordinate**, or **ordinate**, of Q.

The ordered pair (3,2) is the way in which we express the **coordinates** of Q. Notice that the x-coordinate is written first; it is always written first.

From point R, perpendiculars have been drawn to the x-axis and to the y-axis. Since the x-coordinate of R is -4 and the y-coordinate of R is 5, the coordinates of R are $(-4,5)$. What are the coordinates of S? the coordinates of T?

2. Name the coordinates of A; B; C; D; E; F; G; H. Notice that it is convenient to use graph paper; the grid lines act as convenient perpendiculars.



3. Suppose we start with an ordered pair such as $(-2,3)$ and wish to locate the point corresponding to this pair. We reverse the procedure in **1.** A perpendicular is drawn to the x-axis at the point corresponding to -2, and another perpendicular is drawn to the y-axis at the point corresponding to 3. The point of intersection of these perpendiculars has the coordinates $(-2,3)$.

On graph paper, the procedure is simplified. To locate the point with coordinates $(-2,3)$, we begin at the origin and count 2 units *to the left*, then 3 units *up*. Similarly, to locate the point whose coordinates are $(2,-6)$, we begin at the origin and count 2 units *to the right*, then 6 units *down*.

Draw a pair of coordinate axes and mark the point corresponding to each of the pairs just named. Then mark the point corresponding to $(4,5)$; $(-1,-2)$.

4. On graph paper, prepare a set of coordinate axes (as in the diagram for **2**) and mark the points whose coordinates are given below.

a. (5,1)	**b.** (1,5)	**c.** (5,−1)	**d.** (−1,5)
e. (−5,−1)	**f.** (−1,−5)	**g.** (4,2)	**h.** (−4,−2)
i. (2,4)	**j.** (2,−4)	**k.** (−2,4)	**l.** (−4,2)

5. To locate a point whose coordinates involve fractions, you will have to approximate the proper position. Thus, to locate the point with coordinates $(4\frac{1}{3},2\frac{1}{2})$, you count 4 spaces to the right and approximate the extra $\frac{1}{3}$ space. You also must approximate the $\frac{1}{2}$ space for the second coordinate. Mark the point corresponding to: $(4\frac{1}{3},2\frac{1}{2})$; $(-2\frac{1}{2},3)$; $(-4\frac{1}{2},-2\frac{1}{3})$.

6. Choose any point in the coordinate plane. Is there more than one ordered pair associated with this point?

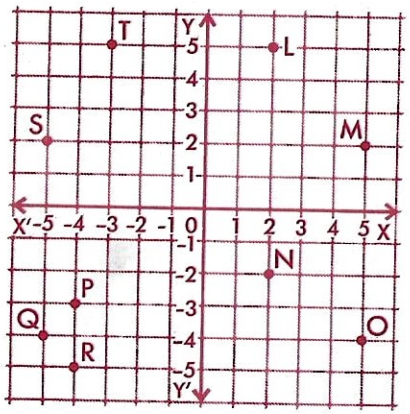
7. Did any two ordered pairs in **4**, page 201, locate the same point? Among the pairs (8,3), (8,−3), (−3,8), (−3,−8), (−8,−3), and (−8,3), are there any two pairs that locate the same point? We state the following generalizations.

▶ To each point in the coordinate plane there corresponds exactly *one* ordered pair of numbers.

▶ To each ordered pair of numbers there corresponds exactly *one* point in the coordinate plane.

Since an ordered pair and its corresponding point in the coordinate plane have this one-to-one correspondence, we often refer to a point by its related number pair. For example, instead of saying "the point P that has coordinates (5,6)," we speak of "the point (5,6)" or simply "$P(5,6)$."

The process of locating and drawing points for a set of number pairs is called **plotting** the set of number pairs. The actual picture is called the **graph** of the set of number pairs.



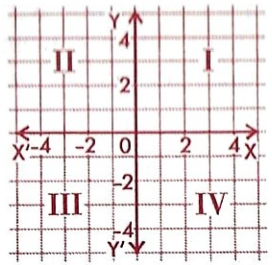
EXERCISES

1. What are the coordinates of L? M? N? O? P? Q? R? S? T?

2. On graph paper, prepare a pair of coordinate axes and plot the points whose coordinates are given below.

a. (1,8)	**b.** $(\frac{1}{2},3)$
c. (−2,−7)	**d.** (−6,1)
e. $(1,-1\frac{1}{2})$	**f.** $(3\frac{1}{4},-2)$
g. $(9,\frac{1}{4})$	**h.** $(\frac{1}{4},-9)$
i. (−3,−8)	**j.** $(-2\frac{1}{4},-2\frac{1}{4})$

Quadrants



The coordinate axes divide the plane into four parts. Each part is called a **quadrant.** The quadrants are named as indicated in the diagram. As you can see, quadrant I is above the x-axis and at the right of the y-axis. In a similar way, describe quadrant II; quadrant III; quadrant IV.

1. Name three points each with an abscissa (x-coordinate) that is positive and an ordinate (y-coordinate) that is positive. Plot these points. In which quadrant do they lie?

2. Name three points each with an abscissa that is negative and an ordinate that is positive. Plot these points. In which quadrant do they lie?

3. In which quadrant will a point be located if its abscissa is positive and its ordinate is negative? In which quadrant will a point be located if its abscissa is negative and its ordinate is negative?

4. Plot the set of points named by these ordered pairs: (5,0), (3,0), $(1\frac{1}{2},0)$, (−1,0), (−4,0). Do they all lie on the x-axis? Will any point with 0 as its ordinate lie on the x-axis?

5. Plot the set of points named by these ordered pairs: (0,2), (0,5), (0,−3), (0,−4). Where are these points located? Will this be true of any point whose abscissa is 0?

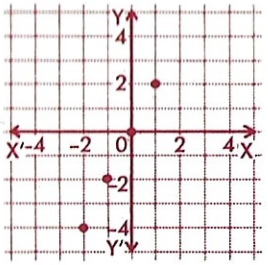
▶ By agreement, a point on either of the axes is not considered to be in any quadrant. Thus, a point such as (3,0) is on the x-axis but not in any quadrant.

6. What is the name of the point (0,0)?

Simple Equality Relations

You know that a relation is a set of ordered pairs. You also know that the coordinates of a point are an ordered pair of numbers. Therefore, we can represent a number relation as a set of points in a coordinate plane. For all of the work that follows, let us agree: Unless otherwise stated, the replacement set for both variables is the set consisting of all the positive numbers, negative numbers, and zero.

1. Let us start with the relation $\{(x,y) \mid y = 2x\}$. You know that this relation consists of all the ordered pairs that are members of the solution set of $y = 2x$.



What replacement for y will make the sentence true if x is replaced by 0? by 1? by 2? by -1?

Is the number pair $(-2,-4)$ a member of the relation? Is $(-3,-6)$? In the figure at the left, we have plotted four of the ordered pairs of the relation. Identify these four. Do they seem to lie on a line?

On graph paper, draw a pair of coordinate axes and plot all six of the number pairs we have found so far. Do they seem to lie on a line? Draw the line. This line is called the *graph of the relation* whose rule is $y = 2x$; that is, the graph of the relation $\{(x,y) \mid y = 2x\}$.

Draw arrows on the line to show that the line extends indefinitely in both directions.

2. Choose a point that is *not* on the graph of $\{(x,y) \mid y = 2x\}$. For instance, is $(3,1)$ on the line? Does the ordered pair $(3,1)$ satisfy the rule $y = 2x$? Pick three other points not on the line. Do their coordinates satisfy the rule $y = 2x$?

3. Consider the relation $\{(x,y) \mid x + 2y = 5\}$. This relation consists of the set of ordered pairs that satisfy the rule $x + 2y = 5$. Can we also say that the relation is the solution set of the open sentence $x + 2y = 5$?

Which of the following ordered pairs satisfy the rule, and which do not?

a. $(1,2)$ **b.** $(3,1)$ **c.** $(4,0)$
d. $(5,0)$ **e.** $(-1,3)$ **f.** $(-3,4)$

Which of the above is the only ordered pair that *is not* a member of $\{(x,y) \mid x + 2y = 5\}$?

4. a. Plot each of the ordered pairs in **3** that is a member of the relation. Do they seem to lie on a line? Draw the line. This is the graph of the relation.

b. Choose several other points that lie on the line. Do their coordinates satisfy the rule $x + 2y = 5$?

c. Now pick three points that *are not* on the line. For each point, test the coordinates to see whether they satisfy the rule $x + 2y = 5$. What do you find?

The graph of a relation tells which ordered pairs are members of the relation. It also indicates which ordered pairs are not members of the relation.

EXERCISES

For each of the following, find four or five number pairs that satisfy the rule. Then draw the figure you think is the graph, and test your figure in the manner described in **4b** and **c**. Remember, the x-replacement is to be the first number in each pair.

1. $\{(x,y) \mid y = x\}$
2. $\{(x,y) \mid x = 2y\}$
3. $\{(x,y) \mid x + y = 5\}$
4. $\{(x,y) \mid y = 2x + 1\}$
5. $\{(x,y) \mid y = x + 5\}$
6. $\{(x,y) \mid x = y + 5\}$
7. $\{(x,y) \mid x - y = 3\}$
8. $\{(x,y) \mid y - x = 3\}$
9. $\{(x,y) \mid y = \frac{1}{3}x\}$
10. $\{(x,y) \mid y = -x\}$

Two Special Cases of Equality Relations

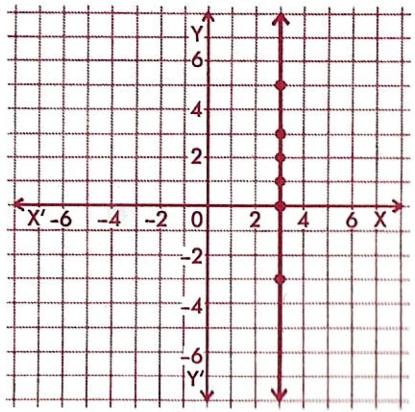
1. Suppose a relation is described as follows:

It is the set of all ordered pairs (x,y) such that the first number in each pair is 3.

Is $(3,0)$ a member of the relation? $(3,1)$? $(3,2)$? $(3,3)$? $(3,-3)$? $(3,5)$?

Why is $(5,3)$ *not* a member of the relation?

Plot the ordered pairs that you decided are members of the relation. Do they lie on a line? Draw the line. You should obtain a figure such as the one shown at the right.



Select three other points on the line. What is the x-coordinate of each? Will every point whose x-coordinate is 3 lie on the line?

Using set notation, we can describe the relation as $\{(x,y) \mid x = 3\}$. We say also that this is a relation whose rule is $x = 3$.

Is the graph of $\{(x,y) \mid x = 3\}$ a line parallel to the y-axis?

2. Name five ordered pairs that are members of the relation $\{(x,y) \mid x = 5\frac{1}{2}\}$. Draw the graph of the relation.

3. Name five ordered pairs that are members of the relation $\{(x,y) \mid x = -1\}$. Draw the graph of the relation.

4. Is each of the graphs you drew for **2** and **3** also a line parallel to the y-axis?

5. Now let us look at the relation described below.

It is the set of all ordered pairs (x,y) such that the second number in each pair is 4.

Is $(4,0)$ a member of this relation? Why not? Is $(0,4)$ a member? $(1,4)$? $(2,4)$? $(4,4)$? $(-2,4)$? $(-4,4)$?

Using set notation, we describe the relation as $\{(x,y) \mid y = 4\}$. The rule of the relation is $y = 4$.

Plot the ordered pairs that you decided were members of the relation. Do they lie on a line? Draw the line. To which axis is this line parallel?

Does every point whose y-coordinate is 4 lie on the line? If a point *is not* on the line, what do you know about its y-coordinate?

6. Name five ordered pairs that are members of the relation $\{(x,y) \mid y = 2\}$; that is, the relation whose rule is $y = 2$. Draw the graph of the relation.

Do you obtain a line parallel to the x-axis?

EXERCISES

Draw the graph of each of the following relations.

1. $\{(x,y) \mid x = 7\}$

2. $\{(x,y) \mid x = -5\frac{1}{2}\}$

3. $\{(x,y) \mid y = -2\}$

4. $\{(x,y) \mid y = \frac{3}{2}\}$

5. $\{(x,y) \mid y = -2x\}$

6. $\{(x,y) \mid x = 3 + 5\}$

7. One of the graphs in Exercises **1–6** *is not* parallel to a coordinate axis. Which graph is this? For each of the others, name the axis to which the graph is parallel.

8. Draw the graph of the relation $\{(x,y) \mid x = 0\}$. What is the common name of this line?

9. Draw the graph of the relation $\{(x,y) \mid y = 0\}$. What is the common name of this line?

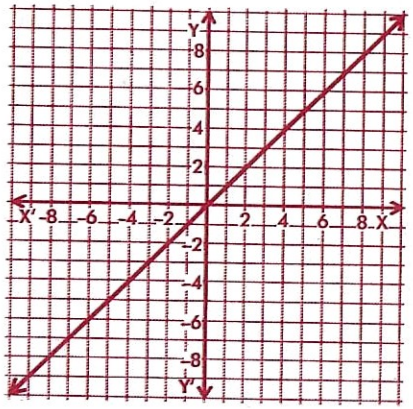
Inequality Relations

1. As has been mentioned, an inequality may be the rule for a relation. Let us investigate the graph of such a relation; for example, the relation $\{(x,y) \mid y \geq x\}$. This is the set of all ordered pairs (x,y) that satisfy $y > x$ *or* $y = x$.

Does $(2,3)$ satisfy $y > x$? Does $(3,3)$ satisfy $y = x$? Then $(2,3)$ and $(3,3)$ are members of the relation $\{(x,y) \mid y \geq x\}$. Why is $(3,2)$ *not* a member of the relation? Which of these are members of the relation $\{(x,y) \mid y \geq x\}$, and which are not?

a. $(6,7)$ **b.** $(6,6)$ **c.** $(6,5)$
d. $(8,9)$ **e.** $(8,8)$ **f.** $(8,7)$

2. Shown here is a graph of all the ordered pairs (x,y) that satisfy $y = x$. Do the pairs $(3,3)$, $(6,6)$, and $(8,8)$ name points on this line?

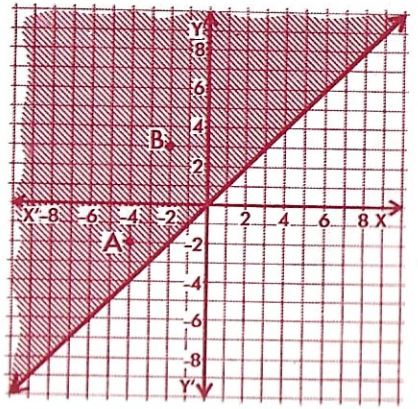


Draw a similar graph and plot the pairs $(6,7)$ and $(8,9)$; that is, two pairs that satisfy $y > x$. Are these points *above* the line?

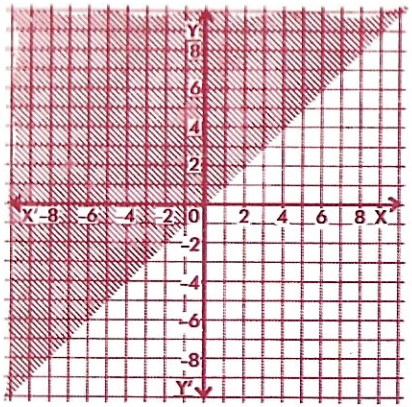
Pick a few more points above the line and test their coordinates. Do they satisfy $y > x$? Then they are members of the graph of the relation $\{(x,y) \mid y \geq x\}$.

Next plot the pairs named in **1c** and **f**; that is, $(6,5)$ and $(8,7)$, which do not satisfy $y \geq x$. Are these points *below* the line?

Do you see that the graph of the relation $\{(x,y) \mid y \geq x\}$ consists of all points on the line and all points above the line? We shade the region of the plane and darken the line to indicate that the line itself is part of the graph. Test the coordinates of the points A and B to see that their coordinates satisfy $y \geq x$.



3. Suppose we were interested in the graph of the relation $\{(x,y) \mid y > x\}$. Do you see that this graph would include only the points *above* the line and *not* those on the line? Still, it is helpful to think of the line associated with $y = x$ in approaching the graph of the relation $\{(x,y) \mid y > x\}$. Similarly, it would be helpful to make use of $y = 2x$ if you were asked to draw the graph of the relation $\{(x,y) \mid y > 2x\}$.

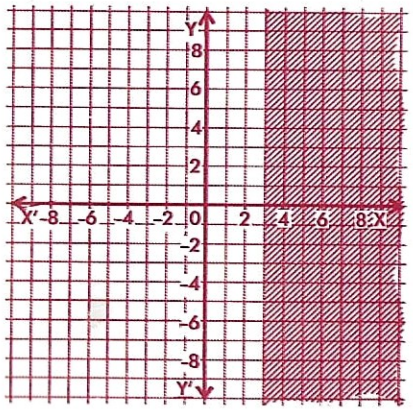


At the left is the graph of the relation $\{(x,y) \mid y > x\}$. Notice that the line associated with $y = x$ *is not* a part of the graph.

Draw the graph of the relation $\{(x,y) \mid y < x\}$.

4. Draw the graph of the relation $\{(x,y) \mid y \leq x\}$.

5. Sometimes it is convenient to interpret relations directly in terms of the coordinate plane. For example, suppose you are asked to plot the graph of the relation $\{(x,y) \mid x > 3\}$. This, as you know, is the set of all ordered pairs (x,y) such that the first member of each pair is greater than 3. We can, however, visualize it easily as the set of all points (x,y) whose x-coordinate is greater than 3.



Look back at **1** on page 205. Then explain why the shaded region shown here is the graph of the relation $\{(x,y) \mid x > 3\}$.

6. Draw the graph of the relation $\{(x,y) \mid x \geq 3\}$.

7. Draw the graph of the relation $\{(x,y) \mid x \leq 3\}$.

EXERCISES

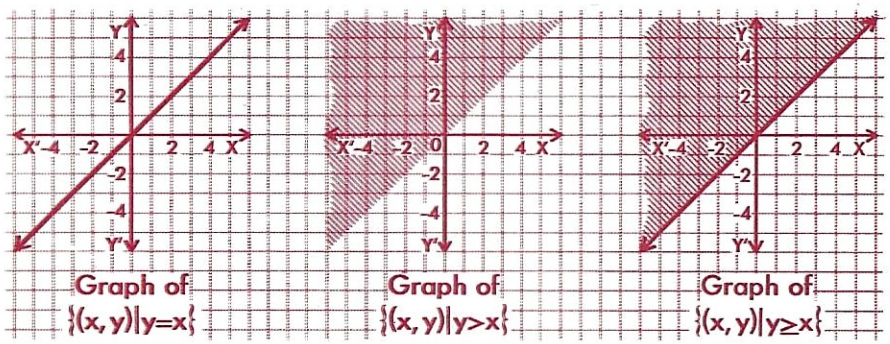
Draw the graph of each of the following relations.

1. $\{(x,y) \mid y \geq 3x\}$
2. $\{(x,y) \mid y > 3x\}$
3. $\{(x,y) \mid y < 3x\}$
4. $\{(x,y) \mid y \leq 3x\}$
5. $\{(x,y) \mid y \leq x + 1\}$
6. $\{(x,y) \mid x \leq 2y\}$
7. $\{(x,y) \mid x > 5\}$
8. $\{(x,y) \mid y \geq 2\}$
9. $\{(x,y) \mid y < 2\}$
10. $\{(x,y) \mid x \leq -2\}$

Union

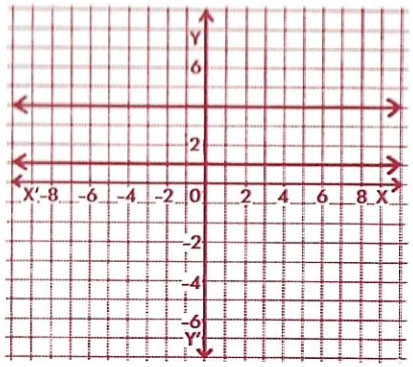
1. Let us take another look at a rule such as $y \geq x$ and the graph of a relation that has this rule. You know that $y \geq x$ can be expressed as a *disjunction*: $y = x$ *or* $y > x$. Does the solution set of $y \geq x$ contain all the ordered pairs (x,y) that are members of the solution set of $y = x$ together with all the ordered pairs that are members of the solution set of $y > x$?

We say that the solution set of $y \geq x$ is the *union* of these two sets. The graph associated with $y \geq x$ can also be described as the union of the graphs of the two sets, as you can see below.



▶ In general, a set C is said to be the **union** of two sets A and B if it contains all the members that belong either to A or to B or to both.

2. Consider the relation $\{(x,y) \mid x \geq 3\}$ in **6**, page 208. Can we describe the solution set of $x \geq 3$ as the union of two sets? Name these sets. Show the union of the graphs in a manner similar to that used above.



3. The union of two sets of ordered pairs is not limited to rules involving "$\geq$" or "$\leq$." For instance, we can graph the union of $\{(x,y) \mid y = 4\}$ and $\{(x,y) \mid y = 1\}$. We draw the graph associated with $y = 4$, then the graph associated with $y = 1$. The union consists of a pair of lines as indicated. What relationship exists between these lines?

4. Draw the graph of the union of the relations $\{(x,y) \mid y = 2\}$ and $\{(x,y) \mid x = -1\}$. The union consists of a pair of lines. Are these lines parallel to each other? If not, what relationship exists between these lines?

Intersection

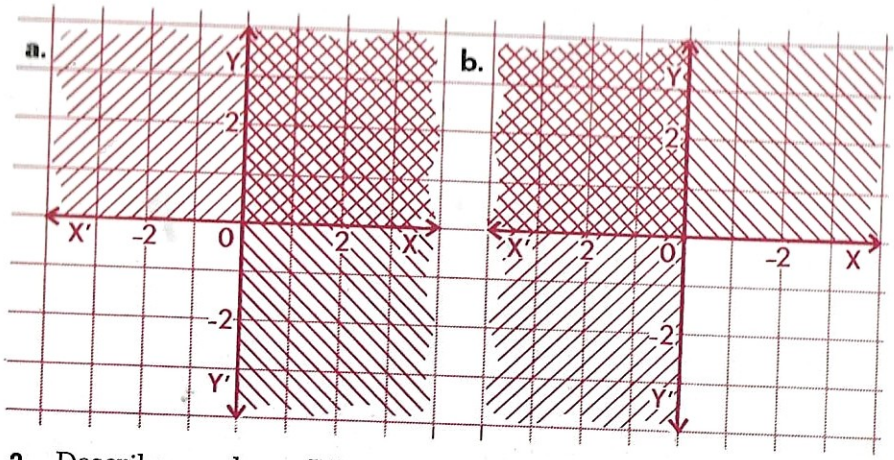
Suppose you are asked to draw a graph of the set of all ordered pairs (x,y) that satisfy the rule $x > 0$ *and* $y > 0$. This rule is, as you know, a *conjunction*. For an ordered pair (x,y) to satisfy $x > 0$ *and* $y > 0$, it must make *both* parts of the sentence true. That is, it must be a pair common to both the solution set of $x > 0$ and the solution set of $y > 0$.

In mathematics, a set C which consists of all members common to two sets A and B is called the **intersection** of sets A and B.

Can we describe the solution set of the sentence $x > 0$ *and* $y > 0$ as the *intersection* of the solution set of $x > 0$ and the solution set of $y > 0$?

1. Draw a graph of the set $\{(x,y) \mid x > 0\}$. On the same pair of axes, draw a graph of the set $\{(x,y) \mid y > 0\}$. Is the intersection of the two graphs a quadrant of the coordinate plane? Which quadrant? Then the relation $\{(x,y) \mid x > 0 \text{ and } y > 0\}$ has quadrant I as its graph.

Which of the graphs on the next page is similar to your graph? Does the other graph represent the sets $\{(x,y) \mid x < 0\}$ and $\{(x,y) \mid y > 0\}$ and their intersection? Which quadrant is represented by the intersection of these graphs? This quadrant, then, may be described as the graph of the relation $\{(x,y) \mid x < 0 \text{ and } y > 0\}$.



2. Describe quadrant III as the graph of a relation.

3. Describe quadrant IV as the graph of a relation.

4. Find the ordered pair or pairs (x,y) that satisfy the rule $y = 2x$ *and* $y = x + 1$. Use the coordinate plane and draw the graphs of $\{(x,y) \mid y = 2x\}$ and $\{(x,y) \mid y = x + 1\}$. How many ordered pairs are members of the intersection of these two sets?

EXERCISES

1. The solution set of $y \leq 5$ can be described as the union of which two sets? Show this relationship by a sequence of graphs as is done on page 209 with $y \geq x$.

2. Draw the graph of the relation that is the union of $\{(x,y) \mid x = 3\}$ and $\{(x,y) \mid x = -4\}$. Can this relation be described as $\{(x,y) \mid x = 3 \text{ or } x = -4\}$?

3. Describe the graph of the relation $\{(x,y) \mid x = 0 \text{ or } y = 0\}$.

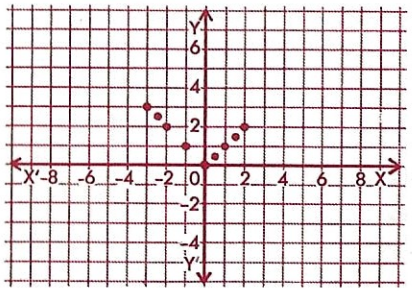
4. Describe the graph of the relation $\{(x,y) \mid x = 0 \text{ and } y = 0\}$.

5. Find the ordered pair or pairs (x,y) that satisfy the rule $y = 3x$ *and* $x = -4$. Use the coordinate plane and draw the graphs of $\{(x,y) \mid y = 3x\}$ and $\{(x,y) \mid x = -4\}$. How many ordered pairs are members of the intersection of these two sets?

Domain and Range

1. Let us investigate the set of all ordered pairs (x,y) such that y is equal to the absolute value of x; that is, the relation $\{(x,y) \mid y = |x|\}$. What does y equal if x is replaced by 2? if x is replaced by $1\frac{1}{2}$? by 1? by $\frac{1}{2}$? by 0?

Suppose now that we replace x by some negative numbers. If x is replaced by -1, what does y equal? if x is replaced by -2? if x is replaced by $-2\frac{1}{2}$? by -3?



2. Do the following ordered pairs you have found satisfy $y = |x|$: $(2,2)$, $(1\frac{1}{2},1\frac{1}{2})$, $(1,1)$, $(\frac{1}{2},\frac{1}{2})$, $(0,0)$, $(-1,1)$, $(-2,2)$, $(-2\frac{1}{2},2\frac{1}{2})$, $(-3,3)$? In this figure, we have plotted these pairs.

Do you see a pattern? Draw a similar figure and plot a few more ordered pairs that satisfy the rule $y = |x|$.

3. No matter what replacement you use for x, will y always be a non-negative number?

In speaking of relations, it is often convenient to use two special words: *domain* and *range*. The set of all first members of the ordered pairs of a relation is called the **domain** of the relation. The set of values that the second member assumes is called the **range** of the relation. Thus, in the case of the relation $\{(x,y) \mid y = |x|\}$ just described, the domain is the set of all positive numbers, negative numbers, and zero; but the range of the relation is the set of non-negative numbers.

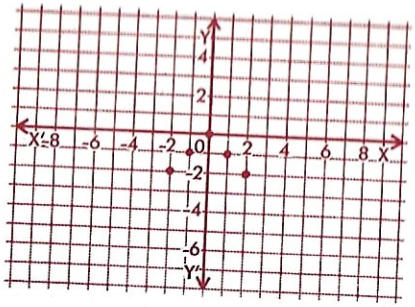
4. Consider the set of ordered pairs (x,y) such that y is equal to the opposite of the absolute value of x; that is, the relation $\{(x,y) \mid y = -|x|\}$.

When x is replaced by 3, does y equal -3? What does y equal when x is replaced by 2? by 1? by 0?

Now let us try some negative numbers as replacements for x. When x is replaced by -1, the absolute value of x is 1. Then y equals the opposite of 1; that is, -1. What does y equal when x is replaced by -2? by -3? by -4?

Five pairs that satisfy $y = -|x|$ have been plotted in this figure. On graph paper, plot these five and at least five others.

The domain of this relation is the set of all positive numbers, negative numbers, and zero. Is the range the set of negative numbers and zero?



5. a. Complete the graph of $\{(x,y) \mid y = |x|\}$.
   b. Complete the graph of $\{(x,y) \mid y = -|x|\}$.

6. Suppose the set $\{1, 2, 3, 4\}$ is the domain for the relation whose rule is $y = 4x$. Is the set $\{4, 8, 12, 16\}$ the range of this relation?

EXERCISES

Draw the graph of each of the following.

1. $\{(x,y) \mid y = |4x|\}$
2. $\{(x,y) \mid y = -|4x|\}$
3. $\{(x,y) \mid y = |x| + 1\}$
4. $\{(x,y) \mid y = 1 - |x|\}$

5. Suppose you wish to draw a graph of the set of ordered pairs (x,y) such that $|x| = 2$. If the absolute value of x equals 2, is it true that $x = 2$ or $x = -2$? Then can the relation $\{(x,y) \mid |x| = 2\}$ be described as the relation $\{(x,y) \mid x = 2 \text{ or } x = -2\}$? Draw the graph of this relation.

6. Draw the graph of the set of ordered pairs (x,y) such that $|y| = 3$.

7. List the ordered pairs in the relation $\{(x,y) \mid x + y = 0\}$ if the domain is the set $\{-5, -4, -3, -2, -1, 0\}$. What set is the range? Plot the points corresponding to the ordered pairs in the relation.

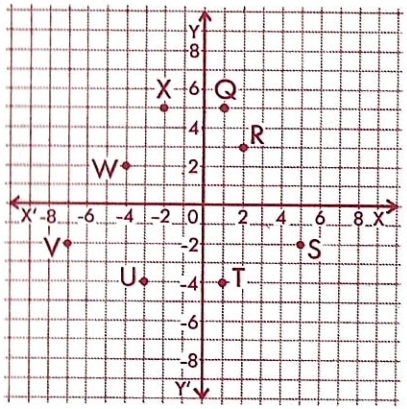
Extra Practice page 403

Challenge page 418

1. Is the ordered pair (2,7) the same as the ordered pair (7,2)? 195–196

2. Study the ordered pairs in each of the following and write a rule that these pairs satisfy. Use x to represent the first number of a pair and y the second number. 197–198

   a. (9,7), (8,6), (7,5), (−1,−3), (−4,−6)
   b. (0,0), (1,4), (2,8), (3,12), (−1,−4)
   c. (0,0), $(1, \frac{1}{2})$, (2,1), (4,2), (−4,−2)



3. In this diagram, what are the coordinates of point Q? R? S? T? U? V? W? X? 200–201

4. Using one pair of coordinate axes, plot each of the following points. 201–203

   a. (−3,−6) b. (3,−6)
   c. (3,6) d. (−1,2)
   e. (−2,−1) f. (6,3)
   g. (−6,3) h. (0,−1)

   Which of the points are in quadrant IV?

5. Which of the following relations have graphs that are parallel to the y-axis? Which have graphs that are parallel to the x-axis? 205–206

   a. $\{(x,y) \mid y = 8\}$ b. $\{(x,y) \mid y = -3\}$
   c. $\{(x,y) \mid x = -4\}$ d. $\{(x,y) \mid x = 20\}$

6. Draw a graph of the relation $\{(x,y) \mid y = x + 3\}$. Which of the following relations would have as its graph the region of the plane *below* the line associated with $y = x + 3$? 207–208

   a. $\{(x,y) \mid y < x + 3\}$ b. $\{(x,y) \mid y > x + 3\}$

7. Suppose the set {1, 2, 3, 4, 5} is the domain of x in the relation whose rule is $y = 3x$. What set is the range of the relation? List all the ordered pairs that are members of the relation and enclose them in braces. 212–213

If you had difficulty with any of the work on this test, find help on the pages listed at the ends of the rows.

LINEAR FUNCTIONS

Description of a Function

A *function* is a special kind of relation. Let us examine two relations, A and B, and make a comparison.

$A = \{(x,y) \mid x = y^2\}$
(4,−2)
(4,2)
(9,3)
(9,−3)

$B = \{(x,y) \mid y = x^2\}$
(4,16)
(9,81)

In relation A, what is the second number when the first is 4? Are there two possible answers? Is there a unique second number when the first is 9?

In relation B, what is the second number when the first is 4? Are there two possible answers? Is there a unique second number when the first is 9? Pick any first number. Does it determine a unique second number?

Relation B is a function. Relation A is not.

A relation is called a **function** if corresponding to each number of the domain there is a unique second number. In other words, each number of the domain occurs only once in the set of ordered pairs.

1. Is the relation $\{(x,y) \mid y = 2x\}$ a function? Try a few x-replacements to see whether each determines a unique second number.

2. Is the relation $\{(x,y) \mid y > 2x\}$ a function? Try a few x-replacements to see whether each determines a unique second number.

EXERCISES

Which of the following relations are functions? Remember: For any first number you use, if you can find more than one second number that will satisfy the rule, the relation is not a function.

1. $\{(x,y) \mid y = x\}$
2. $\{(x,y) \mid y = -x\}$
3. $\{(x,y) \mid y > x\}$
4. $\{(x,y) \mid x + y = 4\}$
5. $\{(x,y) \mid x > 2y\}$
6. $\{(x,y) \mid x + 3y = 0\}$

7. Consider the relation $\{(x,y) \mid x = 2\}$. Are (2,1) and (2,3) members of the relation? Will every ordered pair in this relation have 2 as a first number? Is this relation a function?

The Rule $y = mx + b$

We are now ready to undertake a systematic study of one kind of function. It is a relation whose rule has the form $y = mx + b$. In this rule, $mx + b$ is a polynomial in x, and m and b are any numbers.

From your work in Chapters 3 and 4, you know that m can be referred to as the *coefficient* of x, and b is the *constant term* of the polynomial.

Suppose m is $\frac{1}{2}$ and b is -3. Then the rule $y = mx + b$ is $y = \frac{1}{2}x - 3$, and we have the relation $\{(x,y) \mid y = \frac{1}{2}x - 3\}$.

Express the rule $y = mx + b$ when m is 2 and b is 5.

When m is 3 and b is 0, can we express the rule as $y = 3x$?

What is m if the rule is $y = x + 3$?

The relation $\{(x,y) \mid y = mx + b\}$ is called a **linear function** because, as you will see, its graph is a line.

Let us first prove that a relation with a rule of the form $y = mx + b$ is a function. The rule prescribes a procedure for finding the value of y for any replacement for x. It says that for a given m and b:

(1) multiply the x-replacement by m;

(2) add b to this product.

Is the product obtained in step (1) unique? Why? Is the sum obtained in step (2) unique? Why? Then we see that, for any x-replacement, we obtain a unique number for y. Therefore, $\{(x,y) \mid y = mx + b\}$ is a function for any given m and any given b.

As an illustration of the ideas in the proof, use the rule $y = 3x - 2$ and let the domain of x be the set $\{0, 1, 2, 3, 4\}$. Find the value of y when each of these x-replacements is used. As you work, keep in mind the procedure described in steps (1) and (2) and the questions about uniqueness in the proof.

List the members of the relation $\{(x,y) \mid y = 3x - 2\}$ for the given domain. Does each x-replacement determine a unique second number?

Graphs of Linear Functions

We stated that the graph of $\{(x,y) \mid y = mx + b\}$ is a line for any given m and b. The mathematical proof of this statement is difficult and will be presented in a future course. Meanwhile, we shall try to convince ourselves of the truth of this statement by trying out a number of particular cases.

In each of the following functions, find four number pairs. Plot them. Make a guess as to where the rest of the number pairs in the function are. Draw the figure which you think consists of *all* the points of the function, as far as your graph paper permits. Then perform the following two tests:

(1) Select several points on the figure you drew and see whether the coordinates of each satisfy the rule of the function. They should.

(2) Select several points that are *not* on the figure you drew and see whether the coordinates of each satisfy the rule of the function. They should not.

1. $\{(x,y) \mid y = x + 2\}$ **2.** $\{(x,y) \mid y = 3x\}$

3. $\{(x,y) \mid y = x\}$ **4.** $\{(x,y) \mid y = -3x - 1\}$

5. $\{(x,y) \mid y = x - 2\}$ **6.** $\{(x,y) \mid y = \frac{2}{3}x + 1\}$

7. $\{(x,y) \mid y = 2x + 3\}$ **8.** $\{(x,y) \mid y = -2\}$

9. $\{(x,y) \mid y = \frac{1}{2}x - 2\}$ **10.** $\{(x,y) \mid y = -\frac{2}{5}x + 2\}$

Do you agree that the graph of $\{(x,y) \mid y = mx + b\}$ is a line?

In **1–10**, the domain was not restricted. Therefore, you could draw the lines as far as your paper permitted. Mathematically, each line had no end. As you know, we show this by drawing arrowheads.

11. Let us see what happens if we restrict the domain. For example, suppose the relation is $\{(x,y) \mid y = x + 4\}$ and the domain of x is $0 \leq x \leq 2$. That is, x can be replaced by any number from 0 to 2, inclusive. What does y equal when x is replaced by 0? by 2? Then (0,4) and (2,6) are members of the relation. Find the value that y assumes when x is replaced by $\frac{1}{4}$; $\frac{1}{2}$; 1; $1\frac{1}{2}$; $1\frac{3}{4}$; 1.9.

Plot the points (0,4) and (2,6). Then plot the other pairs you obtained. Draw the graph of the relation $\{(x,y) \mid y = x + 4\}$ for the given domain. Do you think the graph consists of a line segment whose end points are (0,4) and (2,6)?

12. Using the domain $0 \leq x \leq 2$, draw the graph of each of these relations.

a. $\{(x,y) \mid y = 3x\}$ **b.** $\{(x,y) \mid y = x - 1\}$
c. $\{(x,y) \mid y = 2x + 2\}$ **d.** $\{(x,y) \mid y = \frac{1}{2}x\}$

Do you see that if we restrict the domain the graph of the relation $\{(x,y) \mid y = mx + b\}$ is a part of a line?

Expressing Rules as Equivalent Open Sentences

1. Find the number pairs that are members of $\{(x,y) \mid y - x = 2\}$ if the domain is $\{0, 1, 2, 3, 4\}$. Using the same domain, find the number pairs that are members of $\{(x,y) \mid y = x + 2\}$. Are the two sets of number pairs the same?

From your work in Chapters 3–5, you know that, using certain procedures, you can express an open sentence (or a formula) as an equivalent open sentence; that is, as one that has the same solution set as the original sentence. Thus, if the rule of a relation is a sentence such as $y - x = 2$, will we obtain the same set of ordered pairs if we express this rule as an equivalent open sentence?

Is $y = x + 2$ equivalent to $y - x = 2$? Explain how $y = x + 2$ is obtained from $y - x = 2$.

Assume each of the following is the rule for a relation. Try to express each as an equivalent open sentence of the form $y = mx + b$.

2. $2x + y = 8$ **3.** $y - 2x = 7$

4. $x + y = 5$ **5.** $x - 3y = 3$

6. $x - 2y = 8$ **7.** $2x - y = 6$

Study the rules as given in **2–7.** Is each rule of the form $Ax + By = C$, where A, B, and C are numbers? Were you able to express each in the form $y = mx + b$?

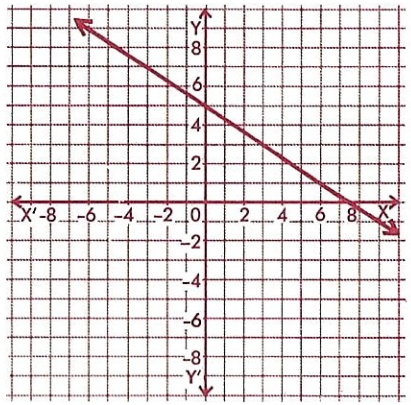
8. Consider the form $Ax + By = C$. If B is 0, the y-term would no longer appear in the rule. Then you *could not* express the rule in the form $y = mx + b$.

For example, if A were 3, B were 0, and C were 9, what rule would you obtain? Can this rule be expressed as the equivalent sentence $x = 3$? Is the relation $\{(x,y) \mid x = 3\}$ a function? (Refer to the discussion on page 215 for help with this question.)

9. Suppose A were 0, B were 3, and C were 9. What rule do you obtain from $Ax + By = C$? Express this rule as a simpler equivalent sentence. Is the relation $\{(x,y) \mid y = 3\}$ a function? If you have answered "No," then give the question more thought. For each x-replacement, is there more than one value for y? Then is this relation a function?

A Note about Procedure

1. To find the graph of the linear function $\{(x,y) \mid 3y + 2x = 15\}$, you might proceed to find some number pairs that satisfy $3y + 2x = 15$. These can be found by trial. For instance, you might use 1 as a replacement for x. Then $3y + 2$ equals 15, $3y$ equals 13, and the value for y must be $4\frac{1}{3}$. Thus, $(1,4\frac{1}{3})$ is a member of the function. Choose three other replacements for x and find the corresponding y-values. Plot the four ordered pairs and draw the graph. Is it like the graph at the right?



2. Often, to find the graph of a function such as $\{(x,y) \mid 3y + 2x = 15\}$, it is easier if the rule is expressed as an equivalent equation of the form $y = mx + b$.

Express $3y + 2x = 15$ in this form. Do you obtain $y = -\frac{3}{2}x + 5$, or do you obtain $y = -\frac{2}{3}x + 5$?

In recording number pairs, we can systematize our work by using a table. Notice that only multiples of 3 were used for x-replacements. What is the advantage of this choice? Plot the points and draw the line. Is it the same as the line you drew for **1**?

x-replacement	−3	0	3	6
y-replacement	7	?	?	?

Do we actually need four number pairs to draw the line? How many points are required to determine a line? On the other hand, using three or four number pairs may help to detect a computational error.

To check the graph itself: Using the original rule $3y + 2x = 15$, you can follow the steps described on page 217.

EXERCISES

Draw the graph of each of the following linear functions. In determining number pairs, you can either use the rule as stated, or you can re-express it in the form $y = mx + b$. However, make a table of values in each case before plotting points.

1. $\{(x,y) \mid x + y = 8\}$
2. $\{(x,y) \mid x - y = 8\}$
3. $\{(x,y) \mid x + y = -8\}$
4. $\{(x,y) \mid x - y = -8\}$
5. $\{(x,y) \mid 2x + y = 4\}$
6. $\{(x,y) \mid -2x + y = 4\}$
7. $\{(x,y) \mid 3x + 2y = 6\}$
8. $\{(x,y) \mid 3x + 4y = 12\}$
9. $\{(x,y) \mid 2y - 3x = 0\}$
10. $\{(x,y) \mid 3x - 4y = 0\}$

11. The open sentences $2x + y = 4$ and $4x + 2y = 8$ are equivalent. Do they have the same solution set? Then will the graph of the relation $\{(x,y) \mid 2x + y = 4\}$ be the same as the graph of the relation $\{(x,y) \mid 4x + 2y = 8\}$?

Without plotting any points, tell whether the following pairs of linear functions have the same graph.

12. a. $\{(x,y) \mid y - x = 15\}$ b. $\{(x,y) \mid y = -x + 15\}$
13. a. $\{(x,y) \mid 2y = 7\}$ b. $\{(x,y) \mid y = 3\frac{1}{2}\}$
14. a. $\{(x,y) \mid 3y - 5x = 15\}$ b. $\{(x,y) \mid y = 2x + 5\}$
15. a. $\{(x,y) \mid 2x - 3y = 6\}$ b. $\{(x,y) \mid y = \frac{2}{3}x - 2\}$

Summary

Let us review some ideas that we have explored.

(1) A function is a relation in which each first number determines a unique second number.

(2) The rule of a linear function is an equation of the form $y = mx + b$, or an equivalent equation.

(3) The graph of a linear function is a line or part of a line.

Therefore, for a given m and b, the equation $y = mx + b$ and all equations equivalent to it correspond to a particular line in the coordinate plane. Such equations are called **linear equations.**

In our work, we will speak of **an equation of a line**, meaning the function which uses the equation as its rule. We do not speak of *the* equation of a line, but *an* equation of a line since many equivalent equations correspond to the same line.

To **plot the graph of an equation**, then, means to plot the relation which has the equation as its rule.

Lines Containing the Origin

1. Using one pair of axes, plot the graph of each equation below. Label the graphs **a**, **b**, **c**, and so on, for convenient reference.

a. $y = x$ **b.** $y = 2x$ **c.** $y = 4x$ **d.** $y = 10x$
e. $y = \frac{3}{4}x$ **f.** $y = \frac{1}{2}x$ **g.** $y = \frac{1}{4}x$ **h.** $y = \frac{1}{10}x$

Study the graphs. Do they all contain the origin? Do you think the graph of $y = 100x$ will contain the origin?

2. Is each of the equations in **1a–h** of the form $y = mx$ where m is a positive number? Study the graphs again, noting the steepness of each line. Which line is the steepest? the least steep?

Can you tell by looking at the coefficient of x which equation will have a steeper line as its graph? How?

3. Do any of the lines in **1** contain points in quadrant II? quadrant IV? Do you think that the graph of $y = 1000x$ will contain the origin and go through quadrants I and III?

4. What is the graph of $y = mx$ if m is zero? Is the line a horizontal line? Can you suggest any coefficient of x that will produce a vertical line? There is none.

5. Using one pair of axes, plot the graph of each equation below. Label the graphs **a**, **b**, **c**, and so on, for convenient reference.

a. $y = -x$ **b.** $y = -2x$ **c.** $y = -4x$ **d.** $y = -10x$
e. $y = -\frac{3}{4}x$ **f.** $y = -\frac{1}{2}x$ **g.** $y = -\frac{1}{4}x$ **h.** $y = -\frac{1}{10}x$

Is each of these equations of the form $y = mx$ where m is a negative number? Do each of their graphs contain the origin? Through which two quadrants does each graph pass?

6. In the graph of $y = x$ (**1a**), if a point were moving on the graph from left to right, would it be rising? Is this true of the graph of each equation in **1**?

Now look at the graphs for equations **5a–h**. From left to right, does the graph of each of these rise, or does it fall?

Do you agree with the following statements about the graph of $y = mx$?

- If m is any number, then the graph of $y = mx$ contains the origin.
- If we "read" the graph from left to right, the line rises when m is a positive number; the line falls when m is a negative number. The line is horizontal when $m = 0$. No matter what m is, the graph of the equation $y = mx$ cannot be a vertical line.

Families of Parallel Lines

1. Using one pair of coordinate axes, plot the graph of each of the following equations.

a. $y = 3x$ **b.** $y = 3x + 2$ **c.** $y = 3x - 2$ **d.** $y = 3x - 4$

Do you obtain a set of parallel lines?

2. Study the equations in **1**. Are they of the form $y = mx + b$ where m is the same number (namely, 3) but each equation has a different b? Do you think that the graph of $y = 3x + 4$ would be parallel to each of the lines in **1**? the graph of $y = 2x + 3$? the graph of $y = 3x + \frac{1}{2}$?

3. The point at which a graph intersects the y-axis is called the **y-intercept** of the graph. What is the y-intercept of graph **1a**? **1b**? **1c**? **1d**? Do you see a pattern in these examples that enables you to predict what the y-intercept of the graph of $y = 3x + 4$ will be? the y-intercept of the graph of $y = 3x + \frac{1}{2}$? Explain.

4. Plot the graph of each of the following equations, using one pair of coordinate axes.

a. $y = 4x + 3$ **b.** $y = 4x$ **c.** $y = 4x - 2$ **d.** $y = 4x - \frac{1}{2}$

Are these lines parallel? Study the equations. Are they of the form $y = mx + b$ where m is the same number in each case but each equation has a different b?

5. Do you think the graph of $y = 5x + 1$ would be parallel to the graphs in **4**? Do you think the graph of $y = 4x + \frac{1}{2}$ would be parallel to these lines? the graph of $y = 4x + 7$?

6. What is the y-intercept of graph **4a**? **4b**? **4c**? **4d**? Is this pattern consistent with the pattern you observed in answering the questions in **3**?

7. Which of the following equations has a graph parallel to the graph of $y = x$?

a. $y = x + 5$ **b.** $y = -x$ **c.** $y = x - \frac{1}{4}$ **d.** $y = x + .1$

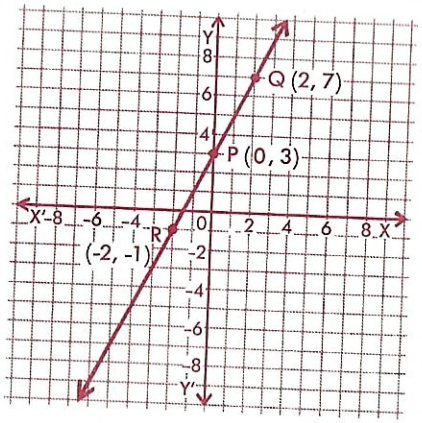
We can summarize as follows:

For any given m and different numbers b_1, b_2, . . . : The equations $y = mx + b_1$, $y = mx + b_2$, and so on, have graphs that are a **family**, or set, of parallel lines. In each case, the number b_1, b_2, . . . , is the y-intercept of the graph.

Slope

You have seen that the graphs of linear functions that have the same m—that is, the same coefficient of x—are parallel. Let us try to find what meaning can be given to m. Let us call m **the slope of the line** that is the graph of $y = mx + b$.

At the right is the graph of $y = 2x + 3$. Then 2 is the slope of the line. Are $P(0,3)$ and $Q(2,7)$ points on the line? Check by seeing whether their coordinates satisfy the equation.



The y-coordinates of these points are 7 and 3. What is their difference? The x-coordinates of these points are 2 and 0. What is their difference?

The quotient of the y-difference divided by the x-difference is $4 \div 2$, or 2. But 2 is also the slope of the line.

Let us take another point, $R(-2,-1)$. Check whether its coordinates satisfy the equation.

What does (y-coordinate of P) − (y-coordinate of R) equal?

What does (x-coordinate of P) − (x-coordinate of R) equal?

Then do we again obtain 2 as the quotient of the y-difference divided by the x-difference?

For any pair of points, let us call the difference between the y-coordinates the **vertical change** and the difference between the x-coordinates the **horizontal change.** Then it seems that:

$$\frac{\text{vertical change}}{\text{horizontal change}} = \text{slope} = m$$

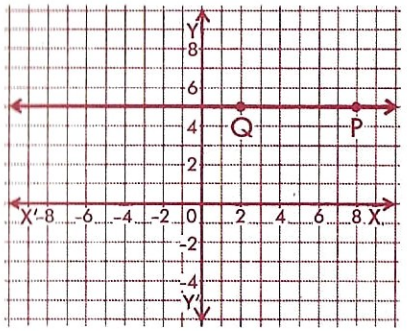
The order in which we consider the points P and Q does not affect the slope. Notice that $\frac{3-7}{0-2} = \frac{-4}{-2}$. The result is still equal to 2.

For each of the following equations, draw the graph and use two ordered pairs (x,y) that satisfy the equation to test whether $\frac{\text{vertical change}}{\text{horizontal change}} = m$.

1. a. $y = \frac{1}{2}x + 2$ b. $y = \frac{3}{4}x$

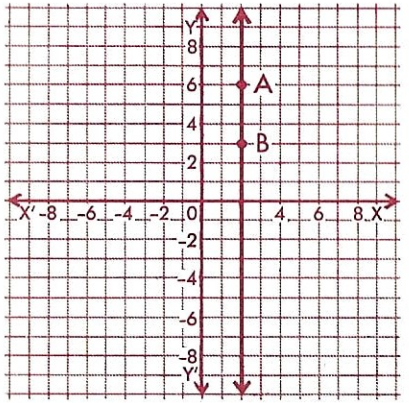
2. a. $y = -\frac{2}{3}x + 1$ b. $y = -6x - 2$

3. On the basis of your work on pages 221–222, how do you interpret a slope that is positive? a slope that is negative?



4. Consider a line that is parallel to the x-axis; for instance, the graph of $y = 5$. What are the coordinates of P? of Q? What is the difference in their y-coordinates? the difference in their x-coordinates? Then the slope is $\frac{0}{6}$, or 0.

Is an equation such as $y = 5$ (or $y = -7$ or $y = \frac{1}{2}$) an equation of the form $y = mx + b$ where m is 0? Then these results are consistent with your earlier work. We say: A line that is parallel to the x-axis has a *zero* slope.



5. Let us now examine the case of a line that is parallel to the y-axis; for instance, the graph of $x = 2$. What are the coordinates of A? of B? What is the difference in their y-coordinates? the difference in their x-coordinates? You know that an expression such as $\frac{3}{0}$ is not defined; that is, division by zero is not permitted.

For any two points on a line that is parallel to the y-axis, would the difference in the x-coordinates always be 0? Then we would always obtain an expression that is not defined.

We say that the slope of a line parallel to the y-axis is not defined. Such a line is said to have *no slope*, not zero slope.

EXERCISES

1. Which of the following equations have graphs that contain the origin?

a. $y = -2x$ b. $y = 5x + .1$ c. $y = -7x$

d. $y = \frac{1}{8}x$ e. $y = 2x + 15$ f. $y = 8x$

g. $y - 4x = 0$ h. $y - 3x + 7 = 0$ i. $y + 3x + 2 = 2$

2. Which of the following equations have graphs that are parallel to the graph of $y = -3x$?

a. $y = 3x + 3$ b. $y = -3x - 2$ c. $y = 3x + 1$

d. $y + 3x = 5$ e. $y = -3x + \frac{1}{2}$ f. $y + 6x = 3x + 1$

Without plotting any points, tell what the slope and the y-intercept are in the graph of each of the following equations.

	a	b	c
3.	$y = 5x - 2$	$y = \frac{1}{3}x + 2$	$y = x - 5$
4.	$y = -x - 5$	$y = -8x$	$y = 9x + .1$

For each of the following equations, write an equivalent equation of the form $y = mx + b$. Then tell the slope and y-intercept of the graph of each.

5.	$x + y = 3$	$2x + y = 7$	$3x - y = 5$
6.	$2x + 3y = 6$	$2y = 12 - 3y$	$3x = 2y$
7.	$5x + 4y = 0$	$5x + 2y = 12$	$y - 2 = 0$

8. Describe the difference between a line whose slope is zero and a line that has no slope.

Models of Functions

Models of functions occur in many areas of life. For example, suppose a train is traveling at a uniform rate of speed of 40 miles per hour. How many miles will the train have traveled after 3 hours? after 6 hours?

1. The number pairs (3, 120) and (6, 240) are members of a relation $\{(t,d) \mid d = 40t\}$ where t represents number of hours and d represents distance in miles.

If t is 9, what does d equal? if t is 12? if t is 24?

Is the rule $d = 40t$ a rule of the form $y = mx$? Then the relation $\{(t,d) \mid d = 40t\}$ is a linear function.

List all the number pairs you have found so far.

Study the pattern of these pairs. When the number of hours is doubled (from 3 to 6, for instance), is the distance doubled? When the number of hours is tripled (from 3 to 9, for instance), is the distance tripled? We say that d **varies directly** as t.

Now consider *any* ordered pair in the set. Will it always be true that the second number in the ordered pair will be 40 times the first number? In the rule $d = 40t$, the number 40 is called the **constant of variation.**

2. For another model of a linear function, we can study the rule $p = .433h$. This formula relates the pressure (p) in pounds per square inch that water has at a depth of h feet. When h is 10, what does p equal? when h is 20? when h is 30? The number pairs (10,4.33), (20,8.66) (30,12.99) are members of the function $\{(h,p) \mid p = .433h\}$. Does p *vary directly* as h? What is the constant of variation?

As you can see, the rule $p = .433h$ may be described as having the form $y = mx$.

3. Referring to the graph of the function $\{(t,d) \mid d = 40t\}$, what interpretation can we give to 40? Referring to the graph of the function $\{(h,p) \mid p = .433h\}$, what interpretation can we give to .433?

4. For the function $\{(t,d) \mid d = 40t\}$, a reasonable domain for t is the set of non-negative numbers. Why? For the function $\{(h,p) \mid p = .433h\}$, is the set of non-negative numbers a reasonable domain for h?

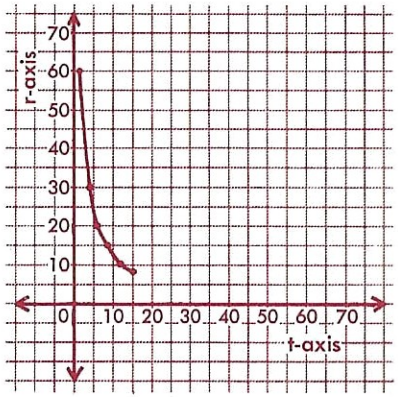
A Non-linear Function

1. To travel a distance of 120 miles in 2 hours, the average speed should be ? miles per hour. To travel 120 miles in 4 hours, the average speed should be ? miles per hour. The two number pairs (2,60) and (4,30) are members of the relation $\{(t,r) \mid rt = 120\}$ where r represents the number of miles per hour and t represents the number of hours.

If t is 6, what is the second number in the ordered pair that is a member of the relation? if t is 8? if t is 12? if t is 15?

List all the number pairs you have found thus far. For each first number, is there a unique second number? Then the relation is a function.

2. On graph paper, draw a pair of coordinate axes. Call the horizontal axis the t-axis; call the vertical axis the r-axis. Let each space on the t-axis represent 5 (hours); let each space on the r-axis represent 5 (miles per hour). Plot the number pairs (2,60), (4,30), (6,20), (8,15), (12,10), (15,8). Do they seem to be on a line? Try to draw a smooth curve that passes through the points. Does it resemble the curve below?



This is an example of a **non-linear function.**

Look at the pattern of number pairs that satisfy the rule. When the number of hours is doubled (from 2 to 4, for instance), is the number of miles per hour halved? When the number of hours is tripled (from 4 to 12, for instance), is the number of miles per hour $\frac{1}{3}$ as great as it was formerly?

Thus, referring to the rule $rt = 120$, we say that r **varies inversely** as t and that the number 120 is the *constant of variation.* You can see that $rt = 120$ can be expressed as $r = 120 \cdot \frac{1}{t}$. (Of course, $t \neq 0$.)

The function $\{(t,r) \mid rt = 120\}$ is an example of a set of functions of the form $\{(x,y) \mid xy = k\}$ where k is a constant. Each such function is a non-linear function.

EXERCISES

1. Draw a graph of the function $\{(h,p) \mid p = .433h\}$ discussed on page 226. Use the horizontal axis as the h-axis. Does any part of the graph appear in quadrant II? quadrant III? quadrant IV?

2. Suppose a \$1000 bond pays 3% interest per year. Then how much interest will the holder of this bond have received after 1 year? after 2 years? after 4 years? The number pairs (1,30), (2,60), (4, 120) are members of the relation whose rule is $I = (.03)(1000)t$. More simply, the rule is stated as $I = 30t$ where t represents the number of years the bond is held and I represents the total amount of interest received in that time. We can denote the relation as $\{(t,I) \mid I = 30t\}$. If we assume that the bond pays interest only yearly, will the domain of t include such numbers as $\frac{1}{2}$, $1\frac{1}{4}$, and so on? What is a reasonable domain for t in this case? Is the relation a function?

3. In the situation involving the train traveling at 40 miles per hour (pages 225–226), can we express the rule $d = 40t$ as $t = \frac{1}{40}d$? Using this form, we might wish to consider the function $\{(d,t) \mid t = \frac{1}{40}d\}$. Is this a linear function? When d is 120, what does t equal? when d is 240? Does t vary directly as d? What is the constant of variation in the re-expressed rule?

4. If y varies directly as x, and (2,8) is one member of the relation, what is the constant of variation?

5. Does the circumference of a circle vary directly as the length of the radius? Write a rule showing this relationship. What is the constant of variation?

6. Using $\frac{22}{7}$ for π, draw a graph of the relation $\{(r,c) \mid c = 2\pi r\}$. Should the domain of r be limited to the set of positive numbers?

7. At a given temperature, the volume (v) of gas in a cylinder varies inversely as the pressure (p) that is applied to it. That is, $v = k \cdot \frac{1}{p}$ (or, stated differently, $p \cdot v = k$ where k is a constant for a specific situation).

If v is 10 when p is 40, we know that the constant is 400, and we have the relation $\{(p,v) \mid v = 400 \cdot \frac{1}{p}\}$. Assume the domain of p is $10 \leq p \leq 40$. Then find eight number pairs that are members of the relation. Express the values of v to the nearest tenth. Then plot these number pairs on a coordinate plane, using the horizontal axis for pressure and the vertical axis for volume. Draw the graph over the interval $10 \leq p \leq 40$. Is this the graph of a linear function?

1. Which of the following relations are functions? Assume that the domain in each is the set of all positive numbers, negative numbers, and zero. 215

Extra Practice Page 404

a. $\{(x,y) \mid y = 5x\}$
b. $\{(x,y) \mid y = 3x + 5\}$
c. $\{(x,y) \mid y > 5x\}$
d. $\{(x,y) \mid y < 5x\}$
e. $\{(x,y) \mid y = 6\}$
f. $\{(x,y) \mid x = 6\}$

2. Draw the graph of the function $\{(x,y) \mid y = -4x\}$. 217

Challenge Page 419

3. Each of the following is a rule for a function. Express each rule as an equivalent open sentence of the form $y = mx + b$. 218

a. $x + 3y = 12$
b. $2x - 3y = 9$
c. $2y - x = 6$
d. $3x + 4y = -8$

Plot the graph of each of these functions. 219–220

4. a. $\{(x,y) \mid x + y = 8\}$ b. $\{(x,y) \mid y = 3x\}$

5. a. $\{(x,y) \mid y = -5\}$ b. $\{(x,y) \mid y = 5\}$

6. a. $\{(x,y) \mid y = 3x - 2\}$ b. $\{(x,y) \mid y = 3x + 2\}$

7. Reading the graph from left to right, is the graph of $y + x = 8$ a line that rises, or is it a line that falls? the graph of $2y - 3x = 7$? 221

8. Name three equations whose graphs are parallel to the graph of $y = -2x$. 222

9. Without plotting, find the slope and the y-intercept of the graph of each of the following equations. 222–224

a. $y = 5x - 2$
b. $y = 2x + 7$
c. $x + y = 16$
d. $3y - 2x = 12$
e. $x - y = 8$
f. $2x + 3y = -3$

10. If oranges cost 8 cents each, is the cost (c) in cents, of n oranges given by the rule $c = 8n$? Assuming that a merchant will not sell fractional parts of oranges, what is a reasonable domain for n in the function $\{(n,c) \mid c = 8n\}$? List five number pairs that are members of the function. Will the graph of the function be a line, or will it be a set of isolated points? Explain. 225–226

If you had difficulty with any of the work on this test, find help on the pages listed at the ends of the rows.

7 Systems of Equations and Inequalities

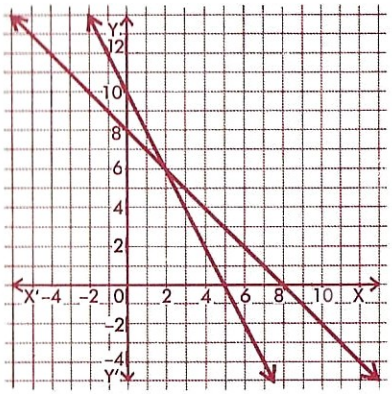
GRAPHS OF SYSTEMS OF EQUATIONS

Union and Intersection of Graphs You have already learned that graphing is an important and useful algebraic tool. Now you are ready to use your knowledge of graphing in new ways.

1. Using x to represent the first number in each case and y the second, express the following as mathematical sentences.

a. Two times a number plus a second number is 10.

b. The sum of two numbers is 8.



2. Study the graphs at the left. Tell which graph represents the solution set of sentence **a** above; of sentence **b**. Using the graphs, name three number pairs that make sentence **a** true; three pairs that make sentence **b** true. Is there any limit to the number of ordered pairs in the solution set of sentence **a**? of sentence **b**?

3. Suppose you were told to consider sentences **a** and **b** as parts of this disjunction: $2x + y = 10$ *or* $x + y = 8$.

You have learned that the solution set of a disjunction is the union of the solution sets of the two parts of the sentence. Does the union of the sets of the above disjunction include every number pair in the solution set of sentence **a**? in the solution set of sentence **b**? Does any point on either graph correspond to a number pair that is in the solution set of the disjunction?

4. Now suppose you consider sentences **a** and **b** as parts of this conjunction: $2x + y = 10$ *and* $x + y = 8$.

Is the solution set of a conjunction the number pair (or pairs) that is a *common* solution to both parts of the sentence—that is, the intersection of the solution sets of the two equations?

Look at the graphs of sentences **a** and **b**. How can you find the intersection of the solution sets of the sentences? How many number pairs make up the intersection of the solution sets? What are the coordinates of the point of intersection?

Replace x and y in the two original sentences by the pair of numbers that name the point of intersection. Are the resulting statements true? Do you think that any other pair of numbers would satisfy both sentence **a** and sentence **b**?

In algebra we are frequently concerned with finding the single ordered pair that satisfies both parts of a conjunction. Often the conjunction is written in the form shown at the right. Notice that a brace is used to indicate that the sentences are to be considered as a conjunction. We call sentences in this form a **system of equations.** The solution set of a system of equations is the intersection of the solution sets of the two equations.

$$\begin{cases} 2x + y = 10 \\ x + y = 8 \end{cases}$$

Graphical Solution of Systems of Equations

1. The first equation in the system at the right is of the form $y = mx + b$. Then is the graph of this equation a straight line? Is the second equation equivalent to an equation of the form $Ax + By = C$? Is its graph also a straight line? In how many points do you think the graphs of the two equations intersect?

$$\begin{cases} y = 2x - 2 \\ 2x = 14 - 2y \end{cases}$$

2. The table at the right shows three number pairs that satisfy the first equation in the system shown above.

x	-1	0	1
y	-4	-2	0

Prepare a second table to show three number pairs that satisfy the second equation.

3. Using your tables, draw the graphs for the system of equations on the same set of axes. Would two ordered pairs be sufficient for determining the graph of each of the equations? Why is it better to obtain the graph by using three ordered pairs?

4. In 1 you agreed that the graph of each equation of the system is a straight line. Does your completed graph confirm your answer? What are the coordinates of the point of intersection of the graphs?

Replace the variables in the system of equations by this number pair. Are both of the resulting statements true? Do you think there is another number pair that is a common solution to both equations? Then does the point of intersection of the graphs correspond to the solution set of the system?

5. Find three ordered pairs that satisfy each equation in this system. Then, on one set of axes, plot the graph of each equation. Use the graphs to find the solution set of the system. Check by replacing the variables in each equation by the number pair you obtain in your solution.

$$\begin{cases} y = 3x + 2 \\ 2x + 3y + 5 = 0 \end{cases}$$

EXERCISES

By graphing, find the solution set of each of the following systems. Check your solutions.

1. $\begin{cases} x - 3 = y \\ x + y = 5 \end{cases}$

2. $\begin{cases} x + y + 1 = 0 \\ y = 3x - 5 \end{cases}$

3. $\begin{cases} x - 2y = 4 \\ 3x - 5y = 8 \end{cases}$

4. $\begin{cases} 2x + 3y = 4 \\ 2x + y = 0 \end{cases}$

5. $\begin{cases} y - 7 = 0 \\ x + y = 2 \end{cases}$

6. $\begin{cases} y - 7 = 0 \\ x + 4 = 0 \end{cases}$

7. $\begin{cases} y = 2x - 6 \\ 3x + 3y = 9 \end{cases}$

8. $\begin{cases} 9x + y + 15 = 0 \\ -x - 4y + 10 = 0 \end{cases}$

Using Systems of Equations to Solve Problems

You have now learned one way of finding a number pair that satisfies a system. The equations in such a system are sometimes called **consistent equations.**

You may notice that it is often possible to solve problems like those in this section by writing an equation in one variable. However, to familiarize yourself further with systems of equations, you are asked to solve all problems in this chapter by first writing a system of equations in two variables.

1. The sum of two integers is 12. When the larger integer is subtracted from twice the smaller, the result is 3. What are the integers?

To solve this problem, first express each of the sentences about the integers as equations. In this case, use x for the larger integer and y for the smaller. Find three number pairs that satisfy each equation. Then plot the graphs of the equations and find their intersection. Finally, check the solution.

2. The sum of two numbers is 11. If 2 times the second is subtracted from the first, the result is 5. What are the numbers?

3. A freight train left the station and traveled eastward at an average speed of 40 miles per hour. Two hours later an express train left the same station, following the freight train on a parallel track at an average speed of 60 miles per hour. How far from the station did the express overtake the freight?

A solution to this problem may be found by using the relationship:

$$\text{time} \times \text{rate} = \text{distance}$$

Does the problem tell how many hours the freight train had traveled before it was overtaken? Then let x represent the number of hours. Does the problem tell how far the freight had traveled when it was overtaken? Let y represent this distance in miles. Does the following equation state the relationship of the rate and travel time to the distance: $40x = y$?

How many hours less than the freight train had the express been traveling when it overtook the freight? If the travel time in hours of the freight is represented by x, can the travel time of the express be represented by $(x - 2)$? When the express overtook the freight, had it traveled the same distance as the freight? Does y represent this distance?

Write an equation that states the relationship of the rate and travel time of the express to the distance y.

You can see that the two sets of conditions in this problem may be expressed by the system of equations at the right. Find the solution set of the system by graphing. Remember that points on the x-axis represent the number of hours of travel and that points on the y-axis represent the number of miles traveled. Since the distances involve larger numbers than the numbers of hours, you may wish to use a different scale on the y-axis than on the x-axis.

$$\begin{cases} 40x = y \\ 60(x - 2) = y \end{cases}$$

4. Use the graphs you drew for **3** to answer the following.
   a. Which train arrives first at a point 200 miles from the station? About how long does it take the train to travel this distance? How far is the other train from the station at the same time?
   b. Which train arrives first at a point 300 miles from the station? About how far from the station is the other train at the same time?

5. A ship cleared port and sailed westward at an average speed of 15 knots. An hour later a second ship cleared the port and sailed westward on the same course at an average speed of 20 knots. How far from port did the second ship overtake the first?

EXERCISES

1. The sum of two numbers is 11. Ten less than twice the larger is two times the smaller. What are the numbers?

2. Two times a number plus a second number is 11. Two times the second number plus the first is 10. What are the numbers?

3. A hiker leaves a lodge and travels along a trail at an average speed of 2 miles per hour. Two hours later a second hiker leaves the lodge and travels after the first hiker at an average speed of 4 miles per hour. How far from the lodge does the second hiker overtake the first?

4. The sum of two numbers is 3. Twice the first added to the second is 2. What are the numbers?

5. A train left a station and overtook a slow freight that had left the same station a half hour earlier. The slow freight had been traveling at an average speed of 30 miles per hour; the faster train, at an average speed of 40 miles per hour. How long did it take the faster train to overtake the freight? How far did each train travel before the freight was overtaken?

Dependent Equations

1. Draw the graphs of the solution sets of each of the equations in the system at the right. Do you find that the graph of each equation lies along the same line? Is there a single number pair that satisfies both equations? Are all the members of the solution set of one equation also members of the solution set of the other?

$$\begin{cases} 3x - 5y + 4 = 0 \\ 9x + 12 = 15y \end{cases}$$

2. Without drawing a graph, it is possible to tell when both equations in a system have identical solution sets. For example, consider the equations given in **1**.

Is the second equation equivalent to:	$9x - 15y + 12 = 0$
May this equation be rewritten as:	$3(3x - 5y + 4) = 0$
If we multiply both members by $\frac{1}{3}$, do we obtain:	$3x - 5y + 4 = 0$

You can see, then, that the two equations in the system are equivalent and, as you learned earlier, equivalent equations have the same solution set.

▶ A system of two equivalent equations is called a system of **dependent equations.** The solution sets of the two equations have the same graph, and the solution set of the system has an unlimited number of members.

3. Tell which of the following systems consist of consistent equations and which consist of dependent equations. If possible, give your answers without graphing the equations.

a. $\begin{cases} 4x + 6y + 10 = 0 \\ 2x + 3y + 5 = 0 \end{cases}$ **b.** $\begin{cases} x + 7 = 2y \\ 3x - 6y = -21 \end{cases}$ **c.** $\begin{cases} 2x + y + 4 = 0 \\ x + y = -5 \end{cases}$

d. $\begin{cases} 3x + 2y + 2 = 21 \\ 2x - y = 1 \end{cases}$ **e.** $\begin{cases} 12x - 3y = 14 \\ 1.2x - .3y = 14 \end{cases}$ **f.** $\begin{cases} 4x + y = 10 \\ 2x + \frac{1}{2}y = 5 \end{cases}$

Inconsistent Equations

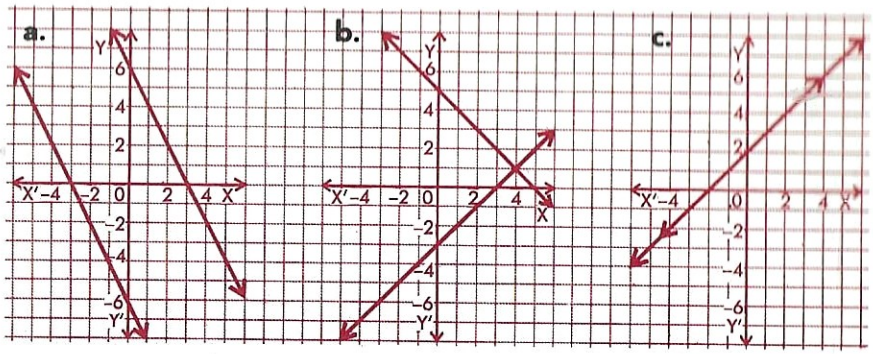
1. Study the system at the right. Are both equations linear? Express each equation in the form $y = mx + b$. Then tell the slope and y-intercept of each equation. (Review pages 222–224, Chapter 6, if necessary.) Will the graphs of the equations be parallel lines? Then will the solution sets of the equations intersect? Is there any number pair that is a common solution to both equations?

$$\begin{cases} 3x - 4y = -7 \\ 3x - 4y = -9 \end{cases}$$

2. Draw the graphs of the solution sets of the system in **1**. Do the graphs confirm the conclusions you reached in **1**?

▶ When the two equations in a system have the same slope but different y-intercepts, they are called **inconsistent equations.** The graphs of the solution sets of the equations are parallel lines, and the solution set of the system is the null set.

3. Which of the following are graphs of consistent equations? of dependent equations? of inconsistent equations?



EXERCISES

Identify the equations in these systems as consistent, dependent, or inconsistent. If possible, decide without drawing a graph.

1. $\begin{cases} 8x = 6y - 2 \\ 4x - 3y = -1 \end{cases}$

2. $\begin{cases} 5x + y = 4 \\ 5x - y - 4 = 0 \end{cases}$

3. $\begin{cases} 3x - 4y = -7 \\ 9x - 12y = -13 \end{cases}$

4. $\begin{cases} 2x + y = 4 \\ y = 2x - 4 \end{cases}$

5. $\begin{cases} y = 3x + 15 \\ y = 3x - 15 \end{cases}$

6. $\begin{cases} \frac{1}{2}x + y - 2 = 0 \\ 2x + 4y = 8 \end{cases}$

SOLVING SYSTEMS BY ADDITION AND SUBSTITUTION

An Experiment with Systems of Equations

1. Look at the equations in the system at the right. Then study the following equation.

$$\begin{cases} 2x + 3y = 13 \\ x - y = -1 \end{cases}$$

$$(2x + 3y) + (x - y) = 13 + (-1)$$

What is the relation between the new equation and the system of consistent equations? Is the expression in the left member of the new equation equal to the sum of the expressions in the left members of the consistent equations? Is this also true of the right member of the new equation and the corresponding members of the consistent equations?

By combining like terms in the new equation, we obtain: $3x + 2y = 12$. Draw graphs of the equations in the system. Then, on the same set of axes, draw a graph of $3x + 2y = 12$.

2. Does the graph of $3x + 2y = 12$ have any points in common with the graphs of the consistent equations? Then does the solution of the system also satisfy $3x + 2y = 12$?

3. Draw graphs for the system at the right. Then write a new equation in which the expression in the left member is equal to the sum of the expressions in the left members of the original equations and in which the expression in the right member is equal to the sum of the expressions in the right members of the original equations. (You will notice that it was necessary to use a great many words in the preceding sentence. From this point on we shall simply say, "Form a new equation by addition," when you are to follow the procedure described in the sentence.) After writing the new equation, simplify it by combining like terms, and then draw its graph. See whether the graph passes through the intersection of the consistent equations.

$$\begin{cases} 2x - 2y = 4 \\ x - 2y = -3 \end{cases}$$

4. Repeat the procedure of **3** for the system at the right.

$$\begin{cases} 3x - 4y = 7 \\ x - y - 3 = 0 \end{cases}$$

5. Look at the third graph you drew for **1**. Is the equation it represents equivalent to either of the two equations in the system? How can you tell? Is the new equation you formed by addition for **3** equivalent to either of the equations in the system?

For any system of consistent equations: If we form a new equation by addition, the new equation is not equivalent to either of the original equations; but the solution of the system also satisfies the new equation.

6. Shown below each of the following systems is its solution set. For each system, form a new equation by addition. Check to see whether the new sentence is true for the pair of numbers in the solution set of the system.

a

$$\begin{cases} x - 2y = 4 \\ 2x + 2y = 14 \end{cases}$$

$\{(6,1)\}$

b

$$\begin{cases} 3y + 2x = 17 \\ y - x = -6 \end{cases}$$

$\{(7,1)\}$

c

$$\begin{cases} 2x + 3y = 4 \\ 2x - 3y = 2 \end{cases}$$

$\{(1\frac{1}{2}, \frac{1}{3})\}$

You have now seen that a new equation formed by addition is satisfied by the solution set of the system. The following explanation may help you to see why this is so.

You know that any linear equation may be expressed as an equivalent equation of the form $Ax + By + C = 0$. Thus, the consistent equations in the system at the right are equivalent to:

$$\begin{cases} 6y = x + 13 \\ 3y = 2x + 8 \end{cases}$$

(1) $6y - x - 13 = 0$
(2) $3y - 2x - 8 = 0$

Then the sum of the expressions in corresponding members of these equations is the following third equation:

(3) $(6y - x - 13) + (3y - 2x - 8) = 0$

Any pair of numbers x and y that makes $6y - x - 13$ equal zero and makes $3y - 2x - 8$ equal zero must also make their sum equal zero. (If $M = 0$ and $N = 0$, then $M + N = 0$.) In other words, any solution for the system of equations must also satisfy the equation formed by addition.

Solving Systems by Addition

From your work with graphs, you know that it is not always possible to read the exact coordinates of a point on the graph. For this reason, the graphical solution of systems is not the most effective method. A more satisfactory method is based on what you have just learned about forming a new equation by addition.

1. For the system at the right, form a new equation by addition. Simplify the new equation. How many variables are there in the new equation? Solve the equation.

$$\begin{cases} 2x - 3y = -1 \\ x + 3y = 13 \end{cases}$$

Use your solution as a replacement for x in the first equation of the system. Solve the first equation for y. You now have found the value of both x and y. Check your solution set by replacing x and y in the second equation of the system.

2. Use the method described above to find the solution sets of these systems. Remember to simplify each equation that you obtain by addition.

a. $\begin{cases} 2x - y = 7 \\ x + y = 8 \end{cases}$ **b.** $\begin{cases} -x + 3y = 9 \\ x + y = 14 \end{cases}$

3. It was possible to find the solution set of each of the systems in **2** because you obtained a coefficient of 0 for one of the variables when you formed the new equation. You can readily see that, if you were to add the expressions in the corresponding members of the consistent equations above, you would not obtain a coefficient of 0 for either of the variables.

$$\begin{cases} 2x + 2y = 20 \\ x + 2y = 14 \end{cases}$$

Let us see whether it is possible to replace one of the equations in the system by an equivalent equation so that we may obtain a coefficient of 0 for one of the variables when we add.

If the term $2y$ in either equation were $-2y$ (the additive inverse of $2y$), what would happen when the y-terms of the two equations were added? If you multiplied both members of the second equation by -1, would $-2y$ be a term in the resulting equivalent equation?

Shown below is the system obtained by multiplying both members of the second equation by -1. Add the expressions in the corresponding members of the equations. Complete the solution by the same methods used in **1**.

$$\begin{cases} 2x + 2y = 20 \\ -x - 2y = -14 \end{cases}$$

4. Find the solution sets of the following by the method used in **3**.

a. $\begin{cases} 2x + y = 12 \\ x + y = 7.5 \end{cases}$ **b.** $\begin{cases} 4x + y = 29 \\ 2x + y = 19 \end{cases}$

5. By now you may have noticed a way to simplify your work in solving systems such as those in **4**. You know that subtraction involves adding the additive inverse of one number to another. Instead of solving the system at the right by multiplying both members of one equation by -1 and then adding, could you solve it by writing the following equation, completing the indicated subtraction, and then finding the solution?

$$\begin{cases} 4x + y = 17 \\ 3x + y = 14 \end{cases}$$

$$(4x + y) - (3x + y) = 17 - 14$$

Try it and see.

It is easier to talk about the subtraction method of solving a system of equations than about "the additive inverse method." So from this point on we shall refer to solutions of consistent equations **by addition** or **by subtraction.** As you have seen, however, both methods actually involve addition.

Two Methods of Writing Solutions

Shown below are two different ways of writing the steps involved in solving a system of equations. The final steps for each method are the same. Method 2 is one that many people consider easier to use, but both are equally correct. Study the model solutions carefully.

$$\text{Solve: } \begin{cases} 7x + y = 19 \\ -y = 8 - x \end{cases}$$

Method 1

Step 1. Form a new equation by addition (or, in some cases, by subtraction).

$$(7x + y) + (-y) = 19 + (8 - x)$$

Step 2. Simplify and solve the new equation.

$$\begin{aligned} 7x &= 27 - x \\ 8x &= 27 \\ x &= \tfrac{27}{8}, \text{ or } 3\tfrac{3}{8} \end{aligned}$$

Step 3. Replace x in one equation by $3\frac{3}{8}$ and solve. (Let us use the second equation, since the solution does not require much computation.)

$$\begin{aligned} -y &= 8 - \tfrac{27}{8} \\ -y &= 4\tfrac{5}{8} \\ y &= -4\tfrac{5}{8} \end{aligned}$$

Check by replacing x and y in the other equation by $\{(3\frac{3}{8}, -4\frac{5}{8})\}$.

Method 2

Step 1. Rewrite one equation so that its terms are in the same position as the like terms in the other equation. Draw a line under the system and add (or, in other cases, subtract) like terms.

$$\begin{array}{r} 7x + y = 19 \\ x - y = 8 \\ \hline 8x \qquad = 27 \end{array}$$

Step 2. Solve the resulting equation.

$$x = \tfrac{27}{8}, \text{ or } 3\tfrac{3}{8}$$

Step 3. Same as for Method 1.

EXERCISES

Solve the following, some by addition and some by subtraction.

1. $\begin{cases} 2x - y = 7 \\ x + y = 11 \end{cases}$
2. $\begin{cases} 2x + y = 17 \\ x = y + 3 \end{cases}$
3. $\begin{cases} x + y = 9 \\ x = 1 - 2y \end{cases}$
4. $\begin{cases} x + y = 4\frac{1}{2} \\ 2x - y = -\frac{3}{4} \end{cases}$
5. $\begin{cases} 3x + y = 25 \\ y = 2\frac{1}{3} - x \end{cases}$
6. $\begin{cases} 2x + y = 22.5 \\ y = x - 3 \end{cases}$
7. $\begin{cases} 2x + 3y - 5 = 0 \\ 4x - 3y + 2 = 0 \end{cases}$
8. $\begin{cases} 2x + 2y = 14 \\ 2x - 3y = 1 \end{cases}$
9. $\begin{cases} 3x - y = 7 \\ 4x + y = 2 \end{cases}$

Solving Systems by Using Equivalent Equations

1. On pages 238–240 you saw how a system of equations could be solved by addition. In using the "additive inverse" method, we formed an equivalent equation by multiplying both members by -1. Let us make further use of equivalent equations.

Notice that in the system at the right neither variable is eliminated by the addition method.

$$\begin{cases} 6y = x + 13 \\ 3y = 2x + 8 \end{cases}$$

Is $12y = 2x + 26$ equivalent to the first equation in the system? Do equivalent equations have the same solution set? Then we may use $12y = 2x + 26$ in place of the first equation. Find the solution set of the second system. Is it the solution set of the first?

$$\begin{cases} 12y = 2x + 26 \\ 3y = 2x + 8 \end{cases}$$

Can you make a guess as to why $12y = 2x + 26$ was chosen as a replacement for the first equation in the original system?

2. In the following solution, you can see that multipliers are chosen that will produce equivalent equations in which one pair of corresponding variables has the same coefficient. Why is it necessary that one pair of corresponding variables have the same coefficient?

$$\begin{aligned} 2x + y - 4 = 0 &\longrightarrow \text{multiply by } 3 \longrightarrow 6x + 3y - 12 = 0 \\ 3x - 4y + 5 = 0 &\longrightarrow \text{multiply by } 2 \longrightarrow \underline{6x - 8y + 10 = 0} \\ &\qquad\qquad \text{subtract} \longrightarrow 11y - 22 = 0 \\ &\qquad\qquad \text{solve for } y \longrightarrow y = 2 \end{aligned}$$

Complete the solution by using 2 as a replacement for y in either of the original equations.

Or:

$$\begin{aligned} 2x + y - 4 = 0 &\longrightarrow \text{multiply by } 4 \longrightarrow 8x + 4y - 16 = 0 \\ 3x - 4y + 5 = 0 &\longrightarrow \underline{3x - 4y + 5 = 0} \\ &\qquad\qquad \text{add} \longrightarrow 11x \qquad - 11 = 0 \end{aligned}$$

Solve for x and complete the solution.

EXERCISES

Find the solutions of the following systems.

1. $\begin{cases} 2x = 3y - 7 \\ 4x + y = 8 \end{cases}$ **2.** $\begin{cases} 3x - y = 4 \\ x - 2y + 4 = 0 \end{cases}$ **3.** $\begin{cases} x + 8y = 2 \\ 2x + 10y = 7 \end{cases}$

4. $\begin{cases} 3x = 7y + 2 \\ 6x = 2y - 4 \end{cases}$ **5.** $\begin{cases} 2x + 4 = 3y \\ x = 3y + 9 \end{cases}$ **6.** $\begin{cases} 3x - y + 2 = 0 \\ 6x - 2y + 4 = 0 \end{cases}$

7. $\begin{cases} 3x - 4y + 1 = 0 \\ 4x - 2y + 11 = 0 \end{cases}$ **8.** $\begin{cases} \frac{1}{2}t - 2y - 8 = 0 \\ 2t + \frac{1}{4}y = 0 \end{cases}$

Problems Involving Systems of Equations

The first group of problems are similar to those you solved graphically on page 233. Solve these problems by the addition or subtraction method.

1. The sum of two numbers is 30. The sum of one-half the smaller and twice the larger is 39. What are the numbers?

2. Three times a number plus twice a second number is 18. Twice the first number plus three times the second number is 17. What are the numbers?

3. An airplane leaves an air field and flies due west at an average speed of 180 miles per hour. Two hours later a second plane leaves the same field and follows the first at an average speed of 300 miles per hour. How long will it take the second plane to overtake the first? How far from the field will the plane be overtaken?

4. Here is another type of problem that may be solved by a system of equations.

It cost a manufacturer \$6000 for equipment with which to manufacture a certain article. The cost of manufacturing each article is \$35. If the articles sell at \$55 each, how many articles must the manufacturer sell in order to break even?

Do we know the number of articles the manufacturer must make and sell in order to break even? Let x represent this number.

The "break-even" point is the amount of money at which expenses and sales are equal. Do we know what this number of dollars is? Then let y represent this number of dollars.

You know that the cost per article is \$35. Then how much do x articles cost? You also know that the manufacturer spent \$6000 for equipment. Do his total costs equal the break-even point? Does this equation state the conditions with respect to costs?

$$35x + 6000 = y$$

How much does the manufacturer receive for 1 article? for x articles? Does the amount he receives for x articles equal the break-even point? Does this equation state these conditions?

$$55x = y$$

Can you complete the solution of the problem?

5. A boy spent \$10 for a plastic molding set. Raw materials for each article he molds cost \$0.15. He plans to charge \$0.40 for each article he sells. How many articles must he make and sell in order to break even?

Solving Systems of Equations by Substitution

1. If you knew the y-value of the solution set of the system at the right, could you then complete the solution by using one of the equations to find the value of x?

$$\begin{cases} y = 3x - 7 \\ 2x - 5y + 4 = 0 \end{cases}$$

Now look at the first equation. Does it say that another way to express y is $3x - 7$? Rewrite the second equation, substituting $3x - 7$ for y. Do you obtain this equation: $2x - 5(3x - 7) + 4 = 0$?

Solve this equation for x. Use your solution as the replacement for x. Can you now solve the first equation for y? Check the values for x and y by using the second equation.

2. Let us solve another system by the method of 1.

$$\begin{cases} 3x + y = 19 \\ 4x - 2y = 12 \end{cases}$$

Can you easily express one of the equations as an equivalent equation in which one of the variables stands alone as the left member?

You probably saw at once that the first equation is equivalent to $y = 19 - 3x$. We say we have solved the equation for y **in terms of x.** Since y may be expressed as $19 - 3x$, may we write the second equation as: $4x - 2(19 - 3x) = 12$? Solve this equation for x. Can you then proceed to find the value of y? What is the solution set of the system?

3. The method described in **1** and **2** is known as the **substitution method** of solving consistent equations. Use this method in solving the following.

a. $\begin{cases} x = 3y - 1 \\ 2x + 4y = 8 \end{cases}$ **b.** $\begin{cases} 2x + y - 7 = 0 \\ 5x - 2y = 8 \end{cases}$ **c.** $\begin{cases} x + y + 1 = 0 \\ x - y - 1 = 0 \end{cases}$

4. In each of the above systems at least one variable has a coefficient of 1. The substitution method may also be used when all of the variables in a system have coefficients greater than 1, as at the right. Always study both equations before re-expressing either. Often one equation may be solved more conveniently than the other.

$$\begin{cases} 6x + 4y - 26 = 0 \\ 5x - 7y - 1 = 0 \end{cases}$$

In this system, there is no obvious advantage in choosing either equation except the fact that the coefficients are smaller in the first.

Solve the first equation for y in terms of x. Do you obtain $y = \frac{26 - 6x}{4}$? Can the expression $\frac{26 - 6x}{4}$ be simplified? Do you obtain $\frac{13 - 3x}{2}$? Substituting this for the y in the second equation, we have:

$$5x - 7\left(\frac{13 - 3x}{2}\right) - 1 = 0$$

Complete the solution for x. Then solve the first equation for y.

Do you think the solution for this system could have been found more easily by addition or subtraction than by substitution? You will find that each method has its advantages. But you need experience in using both in order to be able to decide which to use in solving a particular system.

5. Solve each of the following systems by substitution.

a. $\begin{cases} 4x + 2y = 30 \\ 2x + 3y = 29 \end{cases}$ **b.** $\begin{cases} 3x + 3y = 20 \\ 4x - 4y = 4 \end{cases}$ **c.** $\begin{cases} 5x + 4y = 7 \\ 6x - 7y = 32 \end{cases}$

EXERCISES

Solve by substitution.

1. $\begin{cases} 2x + 2y = 22 \\ 3x - 3y = -9 \end{cases}$ **2.** $\begin{cases} 2x + 2y = 12 \\ 3x + 2y = 13 \end{cases}$ **3.** $\begin{cases} 8x = 6y \\ 2x - 5y = -7 \end{cases}$

4. $\begin{cases} 2y - 6 = 5x \\ 2y - 2x = 18 \end{cases}$ **5.** $\begin{cases} 5x + 3y = 1 \\ 4x + 2y = -8 \end{cases}$ **6.** $\begin{cases} 4x + 3y = 11 \\ 3x - 4y = 2 \end{cases}$

Inconsistent and Dependent Systems

1. Try to solve the system at the right by addition or subtraction. Is it possible to obtain a new equation containing only one variable?

$$\begin{cases} y = 3x + 4 \\ 2y - 6x - 8 = 0 \end{cases}$$

Try to solve the system by substitution. After substitution has been made, what happens when you simplify the resulting equation?

Study the system. Are the equations equivalent? Are they dependent?

2. Try to solve this system by the methods you know. Can you find an ordered pair that satisfies both equations?

$$\begin{cases} x = 6 - 2y \\ x + 2y - 4 = 0 \end{cases}$$

Study the equations. Are they inconsistent?

You know that the solution set of a dependent system contains an unlimited number of ordered pairs and that the solution set of an inconsistent system is the empty set. As you saw in **1** and **2**, the methods used for solving systems of equations are meaningless when applied to dependent and inconsistent systems. You can sometimes save yourself unnecessary work by noticing whether equations in a system are dependent or inconsistent before you begin your solution.

Choose Your Method

Some systems are easier to solve by addition or subtraction than by substitution. Sometimes substitution is more convenient. Sometimes either method is equally good. Decide for yourself which method to use in solving the following.

When the variables x and y are used, the solution may be written in the form $\{(2,3)\}$. It is understood that the first number represents x and the second, y. When other letters are used as variables, it is good practice to designate the solution; for example, $\{s = 4, v = 1\}$. Some of the following systems may be inconsistent or dependent. See whether you can identify such systems before attempting solutions.

1. $\begin{cases} 2y - x = 1 \\ x - 3y = 5 \end{cases}$ **2.** $\begin{cases} 7y + x + 1 = 0 \\ x + 10y = 2 \end{cases}$ **3.** $\begin{cases} 6y + \frac{1}{2} = x \\ 2x - y = 12 \end{cases}$

4. $\begin{cases} 2x + 3y = 9 \\ 3x - 2y = 7 \end{cases}$ **5.** $\begin{cases} r = 3(s + 1) \\ 2r + 3s = 9 \end{cases}$ **6.** $\begin{cases} 3r - 4 = 5s \\ 4r + 5s = 8 \end{cases}$

7. $\begin{cases} v = w + 1 \\ 3v - 2w = 0 \end{cases}$ **8.** $\begin{cases} x - 2y = -4 \\ \frac{1}{2}x + y = 12 \end{cases}$ **9.** $\begin{cases} 3x - 4 = 2y \\ 6x + 4 = -7 \end{cases}$

10. $\begin{cases} 7n = 21 \\ 2m - 7 = 7n \end{cases}$ **11.** $\begin{cases} .3x - .2y = 2.5 \\ 6x + 4y = 10 \end{cases}$ **12.** $\begin{cases} \frac{1}{3}x + \frac{1}{4}y = 1 \\ \frac{2}{3}x - \frac{5}{4}y = 1 \end{cases}$

13. $\begin{cases} \frac{2x}{3} = y \\ 3y + 5x = 14 \end{cases}$ **14.** $\begin{cases} \frac{x+4}{2} = y \\ 2x + y = 7 \end{cases}$ **15.** $\begin{cases} \frac{7x-2}{3} - y = 2 \\ x - \frac{3}{7}y = 3 \end{cases}$

Using Systems to Solve Problems

You have already become familiar with some of the types of problems that may be solved by expressing the conditions as a system of consistent equations. The following problems give you additional practice with these types and introduce you to others that may be solved in a similar manner.

1. Mr. Jones has a "regular" checking account at the Smithfield Bank. It costs him $30¢$ per month plus $7¢$ for each check cashed. Mr. Forbes has a "special" checking account at the same bank. He pays a straight charge of $10¢$ per check. One month both men cashed the same number of checks and were charged the same amount. How many checks did each cash, and how much were they charged?

You can see that if x represents the number of checks each man cashed, and if y represents the amount (in cents) each was charged, the conditions that apply to Mr. Jones are stated by the equation $7x + 30 = y$. Write an equation that represents the conditions that apply to Mr. Forbes. Solve the problem by finding the solution set of the system of equations.

2. In another month Mr. Jones cashed twice as many checks as Mr. Forbes, and his charges were $90¢$ more than those of Mr. Forbes. How many checks did each man cash, and what was each charged?

3. Three apples and 5 oranges cost $42¢$. Seven apples and 9 oranges cost $82¢$. How much does 1 apple cost? How much do 2 apples and an orange cost?

4. Two times a number is 5 more than 3 times another number. The sum of the two numbers is 30. What are the numbers?

5. In a city parking lot the charges are a certain amount for the first hour of parking and a smaller amount for each additional hour. It costs a total of \$2.15 for 5 hours of parking; \$1.80 for 4 hours. What is the fee for the first hour? for each additional hour?

6. The length of a rectangle is three times its width. The perimeter of the rectangle is 20 inches. How wide is the rectangle?

7. A man invested \$3500, part of it at a yearly interest rate of 4% and the rest at 5%. He received a total annual interest of \$153. How much did he invest at each rate? Let x and y represent the two amounts invested. Write an equation that shows the sum of x and y. How do you represent the interest on x dollars at 4%? on y dollars at 5%? Write an equation that shows their sum.

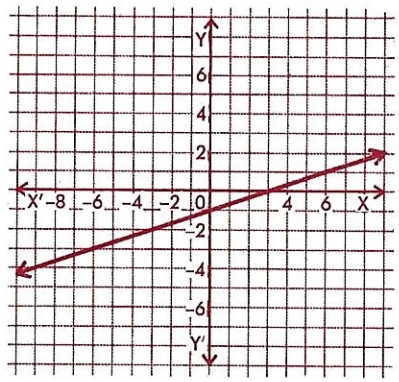
8. A sum of \$2200 is invested, part at 5% and the remainder at 7%. The annual interest on the 5% investment is \$130 less than the interest on the 7% investment. How much was invested at each rate?

9. A police car is ordered to chase a speeding car that is 5 miles ahead. The car is traveling at an average speed of 80 miles per hour, and the police car pursues at an average speed of 100 miles per hour. How long does it take the police car to overtake the car?

SOLVING SYSTEMS OF INEQUALITIES

Points above and below a Line

In Chapter 6 you learned how the graph of an equation such as $x - 3y = 3$ is used in drawing the graph of an inequality such as $x - 3y > 3$. Let us investigate this relationship further.



1. The line in the coordinate plane is the graph of $x - 3y = 3$. Let us express it as an equivalent equation in the form $Ax + By + C = 0$:

$$x - 3y - 3 = 0$$

Choose a point on the graph; for example, (6,1). When the coordinates of this point are used as replacements for x and y in the expression $x - 3y - 3$, do you obtain a value of 0? Do you think that any point on the line has coordinates that give a value of 0 when used as replacements for x and y in $x - 3y - 3$? Verify your conclusion by experimenting with several number pairs.

2. On the coordinate plane, locate the point corresponding to each of the following number pairs. Is each point located above, or below, the graph of $x - 3y - 3 = 0$?

a. $(4,0)$ **b.** $(2,-1)$ **c.** $(0,-2)$ **d.** $(-1,-2)$ **e.** $(7,1)$

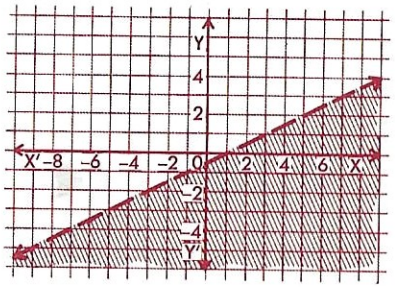
3. Evaluate the expression $x - 3y - 3$, using each of the ordered pairs given above. Do you obtain a positive number in each case? Then is this sentence true for each pair of numbers: $x - 3y - 3 > 0$? Do you think this inequality is true for the number pairs corresponding to every point below the line? Then may we consider all of the points below the line as the graph of $x - 3y - 3 > 0$? Is $x - 3y - 3 > 0$ equivalent to $x - 3y > 3$?

4. Choose several points on the plane above the graph of $x - 3y - 3 = 0$. Use the coordinates of these points to evaluate $x - 3y - 3$. Do you obtain a negative number in each case? May you conclude that the coordinates of all points above the line satisfy this sentence: $x - 3y - 3 < 0$? Then does the graph of this sentence consist of all points above the line that is the graph of $x - 3y - 3 = 0$? Is $x - 3y - 3 < 0$ equivalent to $x - 3y < 3$?

You can see that the graph of a linear equation $Ax + By + C = 0$ divides the coordinate plane into three sets of points:

1. The points on the line that is the graph of the equation
2. All points in the half-plane on one side of the line
3. All points in the half-plane on the other side of the line

All points in one half-plane are the graph of the solution set of $Ax + By + C > 0$; all points in the other half-plane are the graph of the solution set of $Ax + By + C < 0$.



Graphs of Inequalities

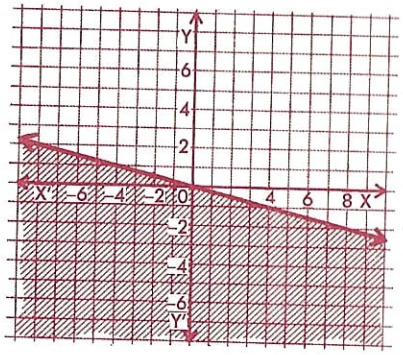
1. To draw the graph of $x - 2y - 1 > 0$, begin with the graph of the equation $x - 2y - 1 = 0$.

Notice that the graph of the equation is shown as a dashed line to indicate that it is not part of the graph of $x - 2y - 1 > 0$. However, it is helpful to indicate the graph of the equation in this way.

Once the graph of the equation is drawn, you must decide in which half-plane to draw the graph of the inequality. Pick any point not on the line, such as $(1,1)$. Replace the variables in $x - 2y - 1$ by these numbers. Do you obtain a negative number? Then the graph of $x - 2y - 1 > 0$ is in the other half-plane.

The completed graph of the inequality is shown on page 248.

2. To draw the graph of the inequality $x + 4y \leq 0$, first draw the graph of $x + 4y = 0$. Since the line is part of the solution, it is not dashed. Next, choose a point not on the line; for example, the point whose coordinates are $(2,2)$. For these replacements, $x + 4y$ is positive. So the remaining points of the graph of $x + 4y \leq 0$ are the region on the other side of the line. At the right is the completed graph.



EXERCISES

Draw the graph of the solution set of each of the following. Remember to use a dashed line if the line is not part of the graph.

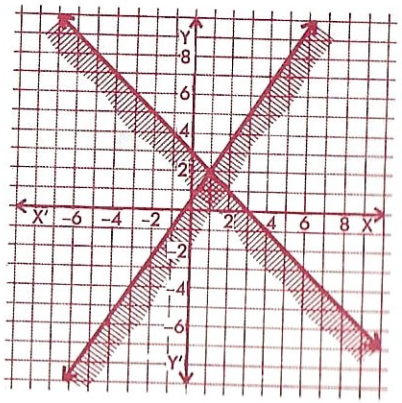
1. $x + 2y > 0$
2. $2x + y - 7 < 0$
3. $-3x + y - 7 \geq 0$
4. $x \leq y + 1$
5. $x + y < 1$
6. $3x - 11y + 8 > 0$
7. $x - 3y + 4 < 0$
8. $2(2x + y) > 0$
9. $2x > 3y - 4$
10. $y \geq 2x + 3$

Graphs of Systems of Inequalities

Graphing provides a convenient method for finding the solution set of a system of inequalities.

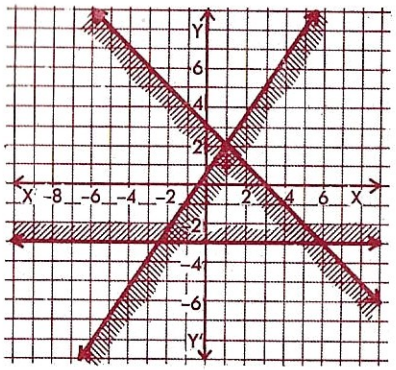
1. You can see that the graphs at the right represent the solution sets of the inequalities in the following system.

$$\begin{cases} 3x - 2y + 1 \geq 0 \\ x \leq 3 - y \end{cases}$$



Up to now, you have shown the graph of an inequality by shading the entire half-plane on one side of the line. Notice how a limited amount of shading is used in the diagram on page 249. The reason for less shading is that it permits easier reading of the graph. However, it is important to remember that the graph of each inequality includes *all* points lying on the side of the line that is shaded.

Which is the graph of the first sentence in the system? of the second? Find the region that represents the intersection of the two graphs. Are points on the lines part of the intersection? Do all points in the intersection have coordinates that satisfy both sentences in the system? Select three different points and verify your answer.



2. On graph paper, copy the graphs shown in **1**. On the same axes, draw the graph of the solution set of $y \geq -3$. Use a limited amount of shading. Does your finished work resemble the diagram at the left?

You have now drawn the graph of the solution set of the following system.

$$\begin{cases} 3x - 2y + 1 \geq 0 \\ x \leq 3 - y \\ y \geq -3 \end{cases}$$

Find the intersection of the graphs of the three sentences in the system. Tell which of the following ordered pairs are members of the solution set of the system.

$(3,3)$ $(1,-2)$ $(1,1)$ $(0,-3)$ $(2,-2)$

EXERCISES

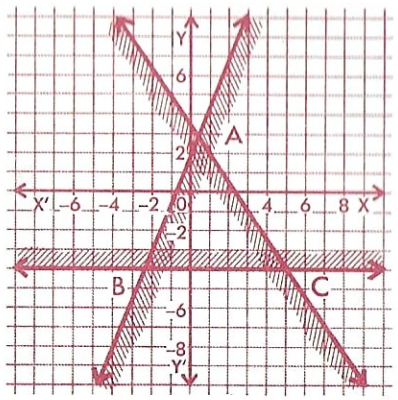
Draw the graph of the solution set of each of the following systems. The same procedure is used for systems of four sentences as for systems of three sentences.

1. $\begin{cases} 3x - 4y + 1 \leq 0 \\ 2y - 7 \leq x \\ y > 1 \end{cases}$ **2.** $\begin{cases} 2x + 5y - 7 \geq 0 \\ 3x - 7y + 1 \leq 0 \\ 3y - 13 \leq 0 \end{cases}$ **3.** $\begin{cases} x \leq y \\ y \leq x + 3 \\ 2 > x + y \\ x + y > -3 \end{cases}$

Finding the Vertices of the Graph of an Inequality System

1. At the right is the graph of the solution set of the following system.

$$\begin{cases} 3x + 2y - 7 \leq 0 \\ 5x - 2y + 4 \geq 0 \\ y \geq -4 \end{cases}$$



You can see that the intersections of the three line graphs are also the vertices of a triangle. Does the graph of the solution set of the system include all points in the area enclosed by the triangle? Does the graph of the solution set also include all points on the line segments that form the triangle? We call the line segments the **boundaries** of the graph of the solution set of the system.

2. It is not easy to tell from the graph what the coordinates are of the intersection of the lines representing $3x + 2y - 7 = 0$ and $5x - 2y + 4 = 0$ (point A). Solve this pair of equations by addition or substitution. Is your solution $(\frac{3}{8}, 2\frac{15}{16})$? Are the two numbers of the solution set the coordinates of point A?

3. Point B is the intersection of the graphs of which two equations? Solve these two equations to find the coordinates of point B.

4. Find the coordinates of point C by solving the two equations whose graphs intersect at C.

It is often convenient to describe the graph of the solution set of an inequality system in terms of the vertices of the geometric figure that is formed. For example, the graph of the system shown above includes all points in the area enclosed by triangle ABC whose vertices have coordinates $(\frac{3}{8}, 2\frac{15}{16})$, $(-2\frac{2}{5}, -4)$, and $(5, -4)$ and all points on the triangle.

5. Suppose the first sentence in the system in **1** is changed to $3x + 2y - 7 < 0$. Are the points on the line segment AC still a part of the graph of the solution set? How would you change the description given above to meet these different conditions?

EXERCISES

By graphing, find the solution set of each of the following systems. Describe the geometric figure that is formed and find the coordinates of the vertices.

1. $\begin{cases} 5y \le 12x + 11 \\ 3x + y \le 13 \\ y \ge -5 \end{cases}$

2. $\begin{cases} 6y \le 7x - 4 \\ 6y + x \ge -20 \\ x \le 4 \end{cases}$

3. $\begin{cases} 5x + y \le 2 \\ 3x - y + 4 \ge 0 \\ 3x + 4y \ge 2 \end{cases}$

4. $\begin{cases} 2x - 3y + 2 \ge 0 \\ 3y + 4 \ge x \\ x - y \le 5 \\ x \le 7 \end{cases}$

5. $\begin{cases} x + y \ge 2 \\ x - y \le 2 \\ 3x - 5y + 4 \ge 0 \end{cases}$

6. $\begin{cases} x - 3y + 5 \ge 0 \\ x + 3y + 5 \ge 0 \\ x \le y \end{cases}$

Linear Programming

Graphs of inequalities are frequently used in industry. The following is a simplified version of the type of problem a mathematician might be asked to solve for a manufacturer. The procedure used in solving the problem is called **linear programming.**

1. A manufacturer makes tables and chairs. Two different machines, A and B, are needed to make each. The manufacture of a table requires 6 hours on machine A and 2 hours on B. The manufacture of a chair requires 1 hour on machine A and 3 hours on B.

The manufacturer makes a profit of \$10 on a table and \$3 on a chair and can sell as many of both as he makes. How many of each should he manufacture in order to make the largest possible profit?

There are two distinct steps in solving the problem. The first step involves making graphs of the conditions that apply to the use of machines A and B.

Let x represent the number of tables and y the number of chairs manufactured in a day. Each machine may run 24 hours a day or less.

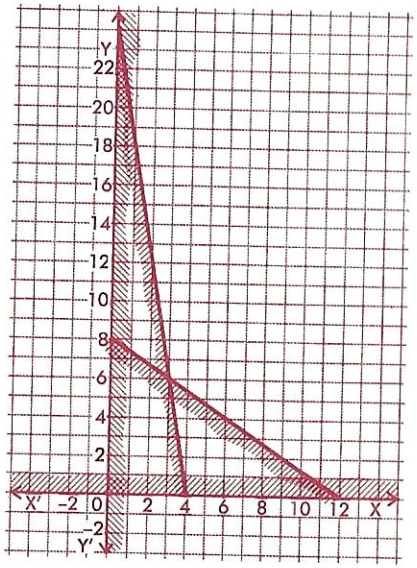
On machine A, one table requires 6 hours of work and one chair requires 1 hour of work. Thus, x tables require $6x$ hours and y chairs require y hours. The total of these two times—that is, $6x + y$—equals the number of hours machine A operates each day. Since the machine can operate 24 hours or less, is the following sentence true for machine A: $6x + y \le 24$?

How much time is required on machine B to manufacture x tables? to manufacture y chairs? Write a sentence that indicates that the number of hours devoted to manufacturing tables plus the number of hours devoted to manufacturing chairs is equal to or less than the number of hours machine B can operate during one day.

You now have a system of two inequalities, but there are two more conditions involved in the situation. Can the number of tables manufactured in one day equal zero? Can the number be less than zero? more than zero? Then may we write $x \ge 0$? Write a similar sentence about the number of chairs that may be manufactured in one day.

Now you have a system consisting of the four inequalities shown below. Draw a graph of the solution set of this system.

$$\begin{cases} 6x + y \le 24 \\ 2x + 3y \le 24 \\ x \ge 0 \\ y \ge 0 \end{cases}$$



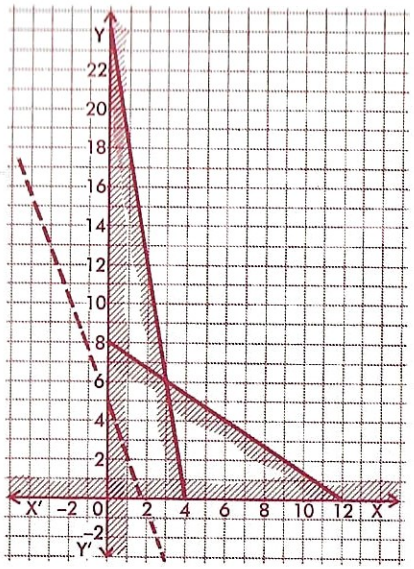
2. The graph you drew for **1** should resemble the one at the right.

Describe the polygon that is the graph of the solution set of the system. What are the coordinates of the vertices?

3. Now let us go on with the second step in solving the original problem.

The profit is \$10 per table and \$3 per chair. Does $10x$ represent the profit on the number of tables made in one day? How would you represent the profit on the number of chairs made in one day? If P stands for the total profit in dollars for one day, may we write the following: $10x + 3y = P$?

Suppose the manufacturer is presently making a profit of \$15 per day. Then does $10x + 3y = 15$? Draw the graph of the solution set of this sentence on the same axes used for the solution set of the system. Use a dashed line for the graph, as shown on the next page.



4. The intersection of the dashed line and the graph of the inequality system shows all the ways in which the manufacturer can make a profit of \$15. Does the point whose coordinates are $(1,1\frac{2}{3})$ appear to be a member of the intersection set? Then if the manufacturer makes 1 table and $1\frac{2}{3}$ chairs, he can make a profit of \$15. (Fractional chairs and tables are possible, since the manufacture can be completed the following day.)

Name two other ways in which the manufacturer can make a profit of \$15.

5. Draw a graph that will show all the possible ways in which the manufacturer can make a profit of \$30 per day.

6. Is it possible for the manufacturer to make a profit of \$60 per day? Try to answer without drawing a graph.

7. We can find the greatest possible profit by continuing to choose different values for P and drawing graphs of $10x + 3y = P$. However, there is a quicker and more convenient method.

You have found that the coordinates of the vertices of the polygon are $(0,0)$, $(0,8)$, $(3,6)$, and $(4,0)$. In turn, use each ordered pair as replacements for x and y in the expression $10x + 3y$ (that is, the profit expression). For which pair do you obtain the greatest value? This value is the largest profit the manufacturer can make in one day, and the replacements for x and y represent the number of tables and chairs that must be made in order to make that profit.

8. Plot the graph of $10x + 3y = 48$ on the same pair of axes on which you drew the graph of the inequality system. Does the line of the graph intersect a vertex of the inequality graph? Would the graph of $10x + 3y = P$, where $P > 48$, intersect the inequality graph at any point? Do you see that 48 is the largest replacement for P that produces a graph that intersects the polygon?

Solving Problems by Linear Programming

1. Because of a sudden drop in the market, the manufacturer mentioned on page 254 finds that he can sell tables for a profit of only \$3 each. To make up the loss, he increases the price of chairs so that he makes a profit of \$5 per chair. If all other conditions are the same as those described in **1**, page 252, how many tables and how many chairs should he make per day for the largest profit?

You can see that the sentence describing the sum of the profit on tables and the profit on chairs is $3x + 5y = P$. Evaluate the expression $3x + 5y$, using the coordinates of each vertex of the quadrilateral. The greatest value you obtain for P is the solution to the problem. If you wish, you may check the solution by graphing.

2. Under the conditions described in the preceding problem, the manufacturer must make no tables and 8 chairs per day in order to clear the largest profit. Will both machines be operated 24 hours a day to make this number of tables and chairs? You can easily tell by replacing x and y by 0 and 8 in the sentence describing the use of machine A, $6x + y \leq 24$, and the sentence describing the use of machine B, $2x + 3y \leq 24$. How many hours per day is each machine used in making no tables and 8 chairs?

3. Because of another change in the market, the manufacturer finds that he is making a profit of \$4 on each table and \$4 on each chair. How many of each should he make to obtain the largest total profit per day? What is the largest profit per day?

4. A small factory makes gidgets and widgets. A gidget requires 3 hours of work on machine A and 2 hours of work on machine B. A widget requires 2 hours of work on machine A and 2 hours of work on machine B. The machines may be operated up to 24 hours a day.

There is a profit of \$2 on each gidget sold and \$3 on each widget. The factory can sell all the gidgets and widgets it can make. How many of each should it make per day to obtain the largest total profit? What is the largest possible daily profit?

Begin your work by writing a system of inequalities and plotting its solution set.

5. Al and Bert have a process for making novelty bottle openers—some gold-plated and some covered with mink. Al's part of the process requires 1 hour for one bottle opener of either type. Bert spends 2 hours on a gold-plated opener and $\frac{1}{4}$ hour on a mink-covered opener. Al can work no more than 5 hours a day, and Bert can work no more than 3 hours a day.

The profit on a gold-plated opener is \$3; on a mink-covered opener it is \$4. How many of each kind should be made per day to obtain the largest possible profit? What is the largest possible daily profit?

On Your Own

The preceding work is intended as an introduction to linear programming. You will meet the topic again in later grades. However, if you wish to experiment further, you can try your hand at the following problems.

1. A construction company intends to build 60 houses within 10 months. The company builds frame houses at the rate of 10 a month and brick houses at the rate of 5 a month. If the profit is \$900 on a wooden house and \$1200 on a brick house, what is the largest profit the company can expect to make, and how many of each kind of house should it build?

2. Al and Bert took Chet into their business. Al's part of the process still requires 1 hour on either type of bottle opener. Bert now spends 2 hours on a gold-plated opener and $\frac{1}{2}$ hour on a mink-covered opener. Chet spends no time on the gold-plated openers and $\frac{1}{3}$ hour on a mink-covered opener. Al can now work up to 10 hours a day; Bert, up to 8 hours a day; and Chet, up to 3 hours a day.

Because of changed market conditions, the profit is now \$4 on a gold opener and \$3 on a mink opener. How many of each type should be made each day in order to obtain the largest profit? What is the largest possible profit?

1. Solve the following systems of equations by graphing. 231–232

a. $\begin{cases} 5x - 4y = 10 \\ 2x + 8y - 4 = 0 \end{cases}$ **b.** $\begin{cases} 2x - 3y + 4 = 0 \\ 2x + y - 4 = 0 \end{cases}$

2. Solve **a** and **b** by addition; **c** and **d** by substitution. 238–241, 243–244

a. $\begin{cases} 3x + 5y + 5 = 0 \\ 4x + 7y + 7 = 0 \end{cases}$ **b.** $\begin{cases} 3x - 5y + 8 = 0 \\ 6x + 5y - 2 = 0 \end{cases}$

c. $\begin{cases} y = 7x - 3 \\ 2x + 3y + 4 = 0 \end{cases}$ **d.** $\begin{cases} 5x - 4y = 10 \\ 2x + 8y - 4 = 0 \end{cases}$

3. Tell which of the following systems is made up of consistent equations; dependent equations; inconsistent equations. 232, 234–235

a

$$\begin{cases} x - 2y + 5 = 0 \\ 3x + 4y - 5 = 0 \end{cases}$$

b

$$\begin{cases} 3x - 2y + 8 = 0 \\ 3x - 6 = 2y \end{cases}$$

c

$$\begin{cases} 4x - 2y = -6 \\ 2x + 3 - y = 0 \end{cases}$$

4. Draw a graph of the solution set: $\begin{cases} x - 2y + 5 > 0 \\ 3x + 4y - 5 < 0 \end{cases}$ 247–250

5. Name the coordinates at which the boundaries of the graph of each of the following systems intersect. 251

a. $\begin{cases} x \geq y \\ 2x + 2y \geq 0 \\ 7x + 2y - 10 \leq 0 \end{cases}$ **b.** $\begin{cases} 5x + 9y \leq 44 \\ -11x + 4y \leq 46 \\ 6x - 13y \geq 29 \end{cases}$

6. The sum of Tim's and Jim's ages now is 16. Ten years from now one-fourth of Tim's age will equal one-fifth of Jim's age. How old is each now? 242

7. A boy is considering going into the business of making lead soldiers and selling them. The machine for making them will cost him \$20, and he figures it will cost an additional $2¢$ per soldier to make them. If he sells the soldiers at $10¢$ each, how many will he have to sell to break even on his business venture? 242–243

8. A workshop produces two models of record players. Model I requires 2 hours of special hand work and 1 hour for final assembly. Model II requires $\frac{1}{2}$ hour of special hand work and $\frac{3}{4}$ hour for final assembly. The workshop operates up to 16 hours a day.

The profit on Model I is \$200 and on Model II \$80. How many of each model should be made per day to obtain the largest possible profit? What is the largest possible profit? 252–254

If you had difficulty with any of the work on this test, find help on the pages listed at the ends of the rows.

Extra Practice Page 405

Challenge Page 420

8 The Set of Real Numbers

RATIONAL AND IRRATIONAL NUMBERS

Expressing Rational Numbers as Decimals From your earlier courses in mathematics, you know that a number such as $\frac{1}{4}$ can be easily expressed as a decimal. Is each of these equal to $\frac{1}{4}$: .25? .250? .2500? Usually, of course, the decimal expression for $\frac{1}{4}$ is given simply as .25.

What is a decimal expression for $\frac{1}{2}$? $-\frac{1}{2}$? $\frac{1}{5}$? $-\frac{1}{5}$?

1. To find a decimal expression for any rational number (such as $\frac{3}{16}$), you can simply perform the indicated division: $16\overline{)3.000000.}$ Find the quotient to six places. Does the quotient "come out even"? Beginning at the fifth place to the right of the decimal point, what does the quotient consist of?

When a decimal expression for a rational number "comes out even"— that is, when it consists only of zeros after a certain place— it is called a **terminating decimal.** Are the decimal expressions for $\frac{1}{2}$ and $\frac{1}{5}$ terminating decimals?

2. To find a decimal expression for $\frac{1}{7}$, perform the indicated division. Carry the quotient out to twelve decimal places; then study the quotient. Is there a particular group of digits which seems to be repeating? What do you believe the thirteenth digit would be? Is it the same as the first? the same as the seventh? What do you suppose the nineteenth digit would be?

$7\overline{)1.000}$

If, in the decimal expression for a number, the same group of digits is repeated again and again in the same order, the decimal is called a **repeating** or **periodic decimal.** A decimal of this sort is often written $.\overline{142857}$ to show that the group of digits 142857 is repeated indefinitely thereafter.

3. Go back and look at the division example in which you expressed $\frac{1}{7}$ as a decimal. Is 3 the remainder after the first step? What is the next remainder? What is each of the next four remainders? Is each of them less than 7? Could you have a remainder greater than or equal to 7? Assuming the division did not "come out even," were there any possible remainders other than the six you found? Once all six possible remainders had been used, the next one had to be a repeat of a previous one. Since you were performing precisely the same operation you performed before (namely, dividing 10 by 7), do you see that the pattern had to repeat itself?

Suppose the denominator had been 13 instead of 7. Would there be twelve possible remainders?

From this discussion, you should see that in the repeating group of digits in a periodic decimal, there will be at most $b - 1$ digits, where b is the denominator of the fraction. There may, however, be fewer than $b - 1$ digits in the repeating group.

4. Express the rational number $\frac{1}{12}$ as a decimal to six places. Notice that the repeating part of the decimal *does not* include the entire decimal. We express $\frac{1}{12}$ as $.08\overline{3}$ to show that just the "3" repeats. Express $\frac{19}{90}$ as a repeating decimal.

5. Rational numbers such as -7 or $\frac{6}{5}$ are easily expressed as -7.0 and 1.2. We speak of these as terminating decimals. (Basically, we are referring to the portion of the numeral to the right of the decimal point.) The number $\frac{19}{6}$ is expressed as $3.1\overline{6}$, and we speak of it as a repeating decimal. Express $\frac{4}{3}$ as a decimal. Is it a terminating decimal, or is it a repeating decimal?

On the basis of your work thus far, do you agree with the following statement?

▶ Any rational number can be expressed as a decimal. The decimal will be either a terminating decimal or a repeating decimal.

Check this statement as you complete the exercises that follow.

EXERCISES

Write each of the following rational numbers as a repeating or terminating decimal.

1. $\frac{1}{9}$	2. $\frac{1}{3}$	3. $\frac{3}{7}$	4. $-\frac{5}{7}$	5. $\frac{2}{9}$
6. $-\frac{7}{9}$	7. $\frac{2}{5}$	8. $\frac{1}{8}$	9. $\frac{51}{99}$	10. $\frac{217}{30}$

Expressing Decimals as Rational Numbers

As you have seen, any rational number can be expressed either as a terminating decimal or a repeating decimal. Let us now investigate whether it is always possible to write a repeating or terminating decimal as a rational number—that is, as the quotient of two integers.

1. Consider the terminating decimal .4532. Can this be expressed as $\frac{4532}{10,000}$? Express .73576 as the quotient of two integers. Express 30.1 as the quotient of two integers. Do you agree that any terminating decimal is a representation of some rational number?

2. Let us now investigate a repeating decimal such as .83333 . . . (that is, $.8\overline{3}$). Let N equal .83333 . . . ; then $10N = 8.3333\ldots$

We now have: $10N = 8.3333\ldots$
and $N = \ .8333\ldots$
Subtracting, we obtain: $9N = 7.5$
or: $N = \frac{7.5}{9}$

Does N equal $\frac{75}{90}$; that is, $\frac{5}{6}$? Therefore, .8333 . . . can be expressed as the quotient of two integers and is a rational number.

Notice the procedure used. The repeating portion of the decimal consisted of one digit, and we multiplied N by 10. If the repeating portion had consisted of two digits, we would have multiplied by 100.

3. As another illustration, let us consider the repeating decimal .732732 . . . (that is, $.\overline{732}$). Let N equal .732732 . . . ; then $1000N = 732.732732\ldots$ (Why have we multiplied by 1000?)

Then: $1000N = 732.732732\ldots$
$N = \ .732732\ldots$
Subtracting, we obtain: $999N = 732$
or: $N = \frac{732}{999}$

Have we shown that .732732 . . . represents a rational number? Explain.

4. Express in fraction form.
   a. .0007 **b.** 4.40009 **c.** .555 . . . **d.** .181818 . . .

On the basis of your work thus far, do you agree with the following statement?

▶ Any terminating or repeating decimal represents a rational number.

EXERCISES

Express each of the following terminating or repeating decimals as the quotient of two integers.

1. .888 . . .	**2.** .3737 . . .	**3.** 3.2
4. 2.777 . . .	**5.** .000865	**6.** 2.4545 . . .
7. .5	**8.** .555 . . .	**9.** $.\overline{3}$
10. $.\overline{67}$	**11.** $.\overline{854}$	**12.** $1.\overline{8}$
13. $1.\overline{35}$	**14.** $2.\overline{405}$	**15.** $.\overline{007}$

Irrational Numbers

You have seen that any rational number can be expressed as a terminating or repeating decimal, and any repeating or terminating decimal is a representation of a rational number. The question then arises: Are there any decimals that do not repeat or terminate? If so, what kind of number does such a decimal represent?

Suppose you were to formulate a decimal numeral by using a spinner in the following way. You spin the arrow, and the digit at which the pointer stops is recorded as the digit in tenths' place. You spin the arrow again, and the digit at which the pointer stops is recorded as the digit in hundredths' place. If you continued this procedure indefinitely, the numeral would be a non-terminating decimal. Do you think it would be a repeating decimal? It seems unlikely that there would be any pattern to the digits, and therefore such a numeral would probably be a non-repeating decimal.

Try the experiment at home and see the results you obtain. If, at any time, you think your numeral is a repeating decimal, spin the arrow again and you will probably break the pattern.

Another way to make it clear that there are non-terminating decimals that are not repeating decimals is to show one. Consider the following:

.0100100010000100000 1 . . .

This numeral uses the digits 0 and 1. First we have a 0 and a 1; then two 0's and a 1; then three 0's and a 1; and so on. You can see this is not a repeating decimal since the pattern keeps changing; at no time can we say that from there on, the decimal will repeat.

1. Is the decimal numeral .10110111011110111110 . . . a non-terminating, non-repeating decimal? Explain how the numeral is formed. What would the next seven digits be?

2. Consider the numeral .12345678910111213141516171819202 1 . . . Look carefully at the digits. Do you see a pattern? What would the next four digits be? Is this a non-terminating, non-repeating decimal?

3. Make up five non-terminating, non-repeating decimals.

You know that a rational number, expressed as a decimal, is either terminating or repeating. Thus, non-terminating, non-repeating decimals cannot represent rational numbers. We need a new name for the numbers represented by non-terminating, non-repeating decimals. Such numbers are called **irrational numbers.** It follows from what we have said that an irrational number *cannot* be expressed as the quotient of two integers.

The set of all rational and irrational numbers taken together is called the set of **real numbers.**

4. Tell why each of the following is a real number.

a. 0 **b.** -1.13 **c.** $\frac{111{,}157}{3{,}000{,}001}$ **d.** .21211211121111 . . .

5. Irrational numbers may be positive or negative. For example, .01001000100001 . . . represents a positive irrational number, and $-.01001000100001$ . . . represents a negative irrational number. Invent two other negative irrational numbers.

EXERCISES

Tell whether each of the following real numbers is rational or is irrational. If you decide it is a rational number, express this number as the quotient of two integers.

1. .322322232222 . . .	2. .75930862	3. $.\overline{3004}$
4. .6565656565 . . .	5. .7878 . . .	6. $7.1\overline{3}$
7. 5.9899100101102 . . .	8. $.567\overline{0}$	9. $1.23\overline{0}$

Are the decimals in Exercises **8** and **9** terminating decimals according to the definition given on page 258? Can they also be considered repeating decimals (with the "0" repeating)? Because of this, we may sometimes refer to such terminating decimals as repeating decimals. In any case, the numbers are rational numbers.

PROPERTIES OF REAL NUMBERS

In Chapter 2 we considered various properties of the set of integers under addition and multiplication; then we considered the properties of the set of rational numbers under these operations. Outlined below are properties that we assume to be true for the set of real numbers. We will not attempt to justify these assumptions. However, they do represent the basis for operations with real numbers, and it is important that you recognize and understand them.

For any real numbers a, b, and c:

$a + b$ is a unique real number	(Addition is closed.)
$a + b = b + a$	(Addition is commutative.)
$(a + b) + c = a + (b + c)$	(Addition is associative.)
$a + 0 = 0 + a = a$	(Zero is the identity element for addition.)
For every a, there is a number $-a$ such that $a + (-a) = 0$	(Every real number has an additive inverse.)

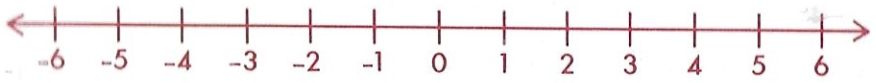
$a \cdot b$ is a unique real number	(Multiplication is closed.)
$a \cdot b = b \cdot a$	(Multiplication is commutative.)
$(a \cdot b)c = a(b \cdot c)$	(Multiplication is associative.)
$1 \cdot a = a \cdot 1 = a$	(One is the identity element for multiplication.)
For every non-zero a, there is a number a^{-1} (that is, $\frac{1}{a}$) such that $a \cdot a^{-1} = 1$	(Every non-zero real number has a multiplicative inverse.)

$a \cdot (b + c) = ab + ac$	(Multiplication is distributive over addition.)

In the set of real numbers, it is also assumed that finding the difference $a - b$ is the same as finding the sum $a + (-b)$; finding the quotient $a \div b$ $(b \neq 0)$ is the same as finding the product $a \cdot \frac{1}{b}$.

Order of Real Numbers

Let us investigate the order of real numbers using a number line. Would you locate .10100100010000 . . . to the left of zero, or to the right of zero? Where would you locate −.10100100010000 . . . ?



1. Consider $3.2\overline{5}$ and 3.254. Which is the smaller number? Then which is located to the left of the other?

2. Is $-3.2\overline{5}$ to the left of −3.255, or to the right of −3.255?

3. Consider the following.

a. $.\overline{1}$ **b.** .10 **c.** $.1\overline{2}$ **d.** .11011101111 . . .

Are all of these numbers between 0 and 1? Which is located farthest to the left on the number line? (That is, which is the smallest number?) If you express **a** as .1111 . . . and **c** as .1222 . . ., you can probably order the numbers more easily.

The number named in **b** is the smallest. Is the number named in **d** smaller than the number named in **a**? (Their first two digits are the same; but the third digit in **d** is "0" and the third digit in **a** is "1," so does it seem reasonable that **d** represents a smaller number than **a**?) Which is the largest of the four numbers?

4. To arrange the numbers $-\frac{9}{8}$, −1.15, $-\frac{8}{7}$, −1.20200200020000 . . . in order from smallest to largest, express $-\frac{9}{8}$ and $-\frac{8}{7}$ as decimals. Which is farthest to the left? Which is next? Which is next? Which is the largest of these numbers—that is, which is farthest to the right?

As you can see from the preceding discussion, any set of real numbers can be ordered from smallest to largest. It is also possible to pair each of these numbers with the points on a line, thus getting what is called the *real number line*. If we were to use only the rational numbers to pair with the points on a line, you can see that some points (those that ought to be paired with irrational numbers) would not have corresponding numbers. Thus, we might say that the rational number line has "holes" in it.

EXERCISES

Order these sets of real numbers from smallest to largest.

1. $\{2.4,\ 2\frac{9}{20},\ 2.\overline{4},\ 2.414114111\ \ldots\}$
2. $\{-2.4,\ -2\frac{9}{20},\ -2.\overline{4},\ -2.414114111\ \ldots\}$
3. $\{\frac{4}{7},\ .545445444\ \ldots,\ .5\overline{4},\ \frac{5}{9},\ -\frac{4}{7},\ -\frac{5}{9}\}$
4. {.010010001 . . . , .001000100001 . . . , −.010010001 . . . , −.001000100001 . . . , 0}

Square Roots

Can you name two numbers each of which has the property that when it is multiplied by itself the result is 4? Is one of these numbers the opposite of the other? Sometimes it is not so easy to name a number which, when multiplied by itself, will yield a certain other number. Therefore, as you know, a special symbol is used to name such numbers. $\sqrt{4}$, read "the square root of four," is that positive number which, when multiplied by itself, results in a product of 4. Thus, $\sqrt{4}\cdot\sqrt{4}=4$. What is the common name for $\sqrt{4}$? You know that there is also a negative number which, when multiplied by itself, results in a product of 4. This number is designated $-\sqrt{4}$. What is another name for this number?

1. What is the positive square root of 25? the negative square root of 25? We write: $\sqrt{25}=5$; $-\sqrt{25}=-5$.

2. In each of the following, replace the question mark with the number that will make the sentence true.

a. $-\sqrt{49}=?$ **b.** $\sqrt{49}=?$ **c.** $\sqrt{121}=?$ **d.** $-\sqrt{100}=?$

An integer such as 16, 25, or 36 is called a **perfect square** since its square roots are integers.

3. Consider $\sqrt{2}$. You know that $1\times 1=1$, $2\times 2=4$, and 2 is between 1 and 4. Then must $\sqrt{2}$ be a number greater than 1 and less than 2? That is, must $\sqrt{2}$ be a number between 1 and 2?

4. Is $\sqrt{3}$ also a number between 1 and 2? Explain.

5. Consider $\sqrt{5}$. You know that $2\times 2=4$ and $3\times 3=9$. Since 5 is between 4 and 9, between which two consecutive integers do you suppose $\sqrt{5}$ is?

6. Since $-\sqrt{5}$ is the additive inverse of $\sqrt{5}$, between which two consecutive negative integers is $-\sqrt{5}$?

EXERCISES

Between which two consecutive integers is each of the following?

1. $\sqrt{7}$
2. $-\sqrt{2}$
3. $\sqrt{19}$
4. $-\sqrt{3}$
5. $\sqrt{14}$
6. $\sqrt{20}$
7. $\sqrt{30}$
8. $\sqrt{40}$
9. $\sqrt{50}$
10. $\sqrt{65}$
11. $-\sqrt{63}$
12. $-\sqrt{82}$
13. $\sqrt{111}$
14. $-\sqrt{150}$
15. $-\sqrt{170}$

Arrange each of these sets of real numbers in order, starting with the smallest.

16. $\{2, -2, \sqrt{2}\}$
17. $\{3, -3, \sqrt{3}\}$
18. $\{0, 5, -5, \sqrt{5}, -\sqrt{5}\}$
19. $\{0, \sqrt{7}, -7, 7, -\sqrt{7}\}$
20. $\{\sqrt{6}, 0, 2, -\sqrt{6}, -2\}$
21. $\{\sqrt{12}, 3, -3, 0, -\sqrt{12}\}$
22. $\{\sqrt{7}, 0, \sqrt{8}, -\sqrt{7}, -\sqrt{8}\}$
23. $\{-\sqrt{13}, -\sqrt{17}, -4, -\sqrt{15}\}$
24. $\{\sqrt{26}, -5, 5, 0, -\sqrt{26}\}$
25. $\{-\sqrt{11}, -\sqrt{10}, 3, -3, \sqrt{11}\}$

Irrationality of Certain Square Roots

1. Let us investigate more closely a number such as $\sqrt{2}$. We have established that $\sqrt{2}$ is between 1 and 2. Or:

$$1 < \sqrt{2} < 2$$

To gain a better idea of the value of $\sqrt{2}$, let us try to find two numbers that are closer together than 1 and 2 and still have $\sqrt{2}$ between them. What does $(1.4)^2$ equal? $(1.5)^2$? Is $\sqrt{2}$ greater than 1.4? Is $\sqrt{2}$ less than 1.5? We write:

$$1.4 < \sqrt{2} < 1.5$$

2. Since $(1.4)^2$ is closer to 2 than $(1.5)^2$, $\sqrt{2}$ is probably closer to 1.4 than to 1.5. Try 1.42 as an approximation of $\sqrt{2}$. What does $(1.42)^2$ equal? You can see that $\sqrt{2}$ must be less than 1.42. Try 1.41. What does $(1.41)^2$ equal? You can see that $\sqrt{2}$ is greater than 1.41. We write:

$$1.41 < \sqrt{2} < 1.42$$

3. From the fact that $(1.41)^2 = 1.9881$ and $(1.42)^2 = 2.0164$, does it seem reasonable that $\sqrt{2}$ is about midway between 1.41 and 1.42? What does $(1.415)^2$ equal? $(1.414)^2$? We write:

$$1.414 < \sqrt{2} < 1.415$$

4. Using the results of **3**, locate $\sqrt{2}$ between two numbers each having four decimal places. Below are two inequalities. Which of these is true?

a. $1.4143 < \sqrt{2} < 1.4144$ **b.** $1.4142 < \sqrt{2} < 1.4143$

No matter how far you carry your work, the number $\sqrt{2}$ cannot be expressed exactly as a terminating or a repeating decimal. The number $\sqrt{2}$ is an irrational number, and the decimal representations we have found are simply *approximations* to $\sqrt{2}$.

It can be shown that for any non-negative integer a that is not a perfect square, $\sqrt{a}$ is an irrational number. Thus, $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{7}$ are irrational numbers. Is $\sqrt{8}$ an irrational number? $\sqrt{9}$? $\sqrt{10}$?

A number such as π is also an irrational number. The rational numbers $\frac{22}{7}$ and 3.14, which you have made use of frequently, are only approximations to π.

EXERCISES

1. Which of the following are irrational numbers and which are rational numbers?

a. $\sqrt{48}$ **b.** $\sqrt{49}$ **c.** $\sqrt{50}$ **d.** $-\sqrt{48}$
e. $-\sqrt{81}$ **f.** $\sqrt{82}$ **g.** $\sqrt{80}$ **h.** $-\sqrt{100}$
i. $-\sqrt{101}$ **j.** $\sqrt{121}$ **k.** $\sqrt{122}$ **l.** $\sqrt{169}$

Which of the following inequalities are true and which are false?

2. $\sqrt{25} < \sqrt{26}$
3. $-\sqrt{25} < -\sqrt{26}$
4. $\sqrt{16} < -\sqrt{25}$
5. $-1 < -\sqrt{2}$
6. $2 < \sqrt{3}$
7. $-3 < -\sqrt{3}$
8. $-\sqrt{5} > -\sqrt{10}$
9. $-\sqrt{2} > -1$
10. $-\sqrt{3} > -2$

Newton's Method

The procedure used on page 266 to find an approximation to $\sqrt{2}$ can be used to find approximations to $\sqrt{3}$, $\sqrt{8}$, and so on.

Another, more systematic method is based on the work of Sir Isaac Newton (1642–1727), the famous English mathematician. We will illustrate this method by finding $\sqrt{28}$.

Does it seem likely that $\sqrt{28}$ will be close to 5? Let us use 5.2 as a guess at $\sqrt{28}$. The first step in the method is to divide 28 by 5.2, carrying the work to one more decimal place than is in the divisor. The completed example is shown on the next page.

If the quotient had been 5.2 exactly, would this have meant that $28 = (5.2)^2$ and that $\sqrt{28} = 5.2$? Since the quotient 5.38 is greater than the divisor 5.2, does this mean that the guess "5.2" was too small?

$$\begin{array}{r} 5.38 \\ 5.2\overline{)\,28.000} \\ 260 \\ \hline 200 \\ 156 \\ \hline 440 \\ 416 \\ \hline 24 \end{array}$$

By taking the average of the original guess and the quotient, we obtain a fairly good approximation to $\sqrt{28}$. That is, $\frac{5.2 + 5.38}{2} = 5.29$.

To find a more accurate approximation to $\sqrt{28}$, simply divide 28 by 5.29 and carry the work to three decimal places. Then take the average of the quotient and the divisor. This average (expressed to three decimal places) will then represent $\sqrt{28}$, correct to the nearest thousandth.

$$5.29\overline{)28.00000}$$

Carry out this procedure. Which of the following is an approximation to $\sqrt{28}$, correct to the nearest thousandth?

a. 5.289 **b.** 5.292 **c.** 5.294

1. Using Newton's method, approximate $\sqrt{7}$ to the nearest hundredth.

2. Using Newton's method, approximate $\sqrt{59}$ to the nearest thousandth.

3. Approximate $-\sqrt{71}$ to the nearest hundredth.

EXERCISES

Approximate to the nearest tenth.

1. $\sqrt{35}$ **2.** $\sqrt{50}$ **3.** $\sqrt{130}$

4. $\sqrt{500}$ **5.** $\sqrt{300}$ **6.** $\sqrt{85}$

7. $\sqrt{800}$ **8.** $\sqrt{1000}$ **9.** $\sqrt{1056}$

Approximate to the nearest hundredth.

10. $\sqrt{24}$ **11.** $\sqrt{8}$ **12.** $\sqrt{37}$

13. $\sqrt{3}$ **14.** $\sqrt{5}$ **15.** $-\sqrt{6}$

16. $\sqrt{72}$ **17.** $-\sqrt{53}$ **18.** $-\sqrt{13}$

19. $\sqrt{177}$ **20.** $\sqrt{253}$ **21.** $-\sqrt{319}$

RADICALS

Powers and Roots

As you know, $\sqrt{7}$ is that positive number such that $(\sqrt{7})^2 = 7$. $\sqrt{7}$ is the positive square root of 7, sometimes referred to as the *principal* square root of 7. (As you have learned, $-\sqrt{7}$ is also a square root of 7.) The symbol "$\sqrt{7}$" is called a **radical**. The number 7 is the **radicand**.

1. When we speak of a **cube root** of a number a, we mean a number b such that $b \cdot b \cdot b = a$. That is, $b^3 = a$. Using the radical symbol, we write: $b = \sqrt[3]{a}$ (read "b is the cube root of a"). It follows that the number $\sqrt[3]{a}$ is a number such that $\sqrt[3]{a} \cdot \sqrt[3]{a} \cdot \sqrt[3]{a} = a$.

Similarly, if we speak of a **fifth root** of a number x, we mean a number y such that $y \cdot y \cdot y \cdot y \cdot y = x$. That is, $y^5 = x$. Using a radical, we write: $y = \sqrt[5]{x}$ (read "y is the fifth root of x"). Is this sentence true: $\sqrt[5]{x} \cdot \sqrt[5]{x} \cdot \sqrt[5]{x} \cdot \sqrt[5]{x} \cdot \sqrt[5]{x} = x$?

From the discussion above, you can see the pattern between *powers* and *roots*. If $m^4 = n$, then m is said to be a **fourth root** of n. Is 2 a fourth root of 16? Is -2 also a fourth root of 16? Just as with the square root symbol, $\sqrt[4]{16}$ denotes the positive fourth root of 16. To denote the negative fourth root of 16, we use $-\sqrt[4]{16}$. Thus, $\sqrt[4]{16} = 2$; $-\sqrt[4]{16} = -2$.

2. Is 3 a cube root of 27? Is -3 also a cube root of 27? If your answer to both questions was "yes," then recheck your work carefully: $3^3 = \underline{?}$; $(-3)^3 = \underline{?}$.

What is the cube root of 27? What is the cube root of -27? What is the cube root of 125?

3. The symbols "$\sqrt[2]{7}$" and "$-\sqrt[2]{7}$" might be used to indicate the square roots of 7—that is, a small "2" might be written with the symbols—but it is customary not to do so. The number 3 is said to be the **index** of $\sqrt[3]{125}$; the number 4 is the index of $\sqrt[4]{16}$; the number 5 is the index of $\sqrt[5]{15}$; and so on. The number 2 is the index of $\sqrt{7}$, even though the "2" is not usually written with the symbol. What is the index of $\sqrt[6]{19}$? What is the radicand? What is the index of $\sqrt{5}$? What is the radicand?

4. What does $(\sqrt{5})^2$ equal? $(\sqrt[3]{11})^3$? $(\sqrt[4]{21})^4$?

5. What is the common name for $\sqrt[3]{64}$? $-\sqrt[3]{64}$? $\sqrt[5]{1}$? $\sqrt[4]{625}$?

Rational Roots

Whenever the radicand of a square root is the square of a rational number, we can express the root as a fraction or an integer (rather than in radical form). You have already had experience with integers that are perfect squares. For example, you know that $\sqrt{16} = 4$ and $\sqrt{121} = 11$. Then what does $\sqrt{16} + \sqrt{121}$ equal?

1. Is the rational number $\frac{4}{9}$ the square of $\frac{2}{3}$? That is, does $(\frac{2}{3})^2$ equal $\frac{4}{9}$? Then $\sqrt{\frac{4}{9}} = \frac{2}{3}$. What does $\sqrt{\frac{1}{4}}$ equal? $\sqrt{\frac{25}{36}}$? $\sqrt{\frac{9}{4}}$?

2. What does $\sqrt{.01}$ equal? $\sqrt{.04}$? $-\sqrt{.09}$?

EXERCISES

Express each of the following as an integer or a fraction.

1. $\sqrt{36}$	2. $-\sqrt{36}$	3. $\sqrt[3]{8}$	4. $\sqrt[4]{81}$
5. $\sqrt{1}$	6. $-\sqrt{16}$	7. $\sqrt{400}$	8. $\sqrt{0}$
9. $\sqrt{\frac{9}{4}}$	10. $\sqrt{\frac{1}{16}}$	11. $-\sqrt{\frac{4}{25}}$	12. $\sqrt{\frac{9}{16}}$
13. $\sqrt{.16}$	14. $\sqrt{.36}$	15. $\sqrt[3]{.027}$	16. $-\sqrt[3]{1}$
17. $\sqrt{.0001}$	18. $\sqrt{.0004}$	19. $\sqrt{\frac{121}{121}}$	20. $\sqrt{\frac{225}{1}}$

Simplifying Radicals

1. a. What does $\sqrt{4 \cdot 25}$—that is, $\sqrt{100}$—equal?
   b. What does $\sqrt{4}$ equal? $\sqrt{25}$? $\sqrt{4} \cdot \sqrt{25}$?
   c. Is the following sentence true: $\sqrt{4 \cdot 25} = \sqrt{4} \cdot \sqrt{25}$?

2. a. What does $\sqrt{9 \cdot 16}$—that is, $\sqrt{144}$—equal?
   b. What does $\sqrt{9}$ equal? $\sqrt{16}$? $\sqrt{9} \cdot \sqrt{16}$?
   c. Is this sentence true: $\sqrt{9 \cdot 16} = \sqrt{9} \cdot \sqrt{16}$?

▶ For all non-negative real numbers a and b, $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$.

3. We can use this principle to help in approximating numbers such as $\sqrt{200}$. As you know, one way to approximate $\sqrt{200}$ with a terminating decimal is to use Newton's method. However, if you know that $\sqrt{2}$, to the nearest thousandth, is 1.414, and you know that $\sqrt{200} = \sqrt{2} \cdot \sqrt{100}$, you can say that $\sqrt{200}$, to the nearest hundredth, is $1.414 \cdot 10$, or 14.14.

4. Using the fact that $\sqrt{3}$, to the nearest thousandth, is 1.732, evaluate $\sqrt{300}$ to the nearest hundredth.

5. Even when we do not wish to approximate a root with a terminating decimal, it is often convenient to *simplify* a radical such as $\sqrt{200}$ by expressing it as $10\sqrt{2}$. A radical is said to be *simplified* if all of the factors of the radicand that are perfect squares have been determined and the radical has been re-expressed so that these factors no longer appear under the radical sign. Thus, to simplify a radical such as $\sqrt{128}$, we proceed as shown at the right.

$$\begin{aligned}\sqrt{128} &= \sqrt{64 \cdot 2} \\ &= \sqrt{64} \cdot \sqrt{2} \\ &= 8\sqrt{2}\end{aligned}$$

Suppose you had written $\sqrt{128} = \sqrt{4 \cdot 32}$, or $2\sqrt{32}$. You would not have finished simplifying the radical, since 32 has a factor of 16, which is a perfect square. You would then have to complete your work as follows: $2\sqrt{32} = 2\sqrt{16} \cdot \sqrt{2}$, or $2 \cdot 4\sqrt{2}$, which is $8\sqrt{2}$.

Simplify each of these radicals: $\sqrt{27}$; $\sqrt{80}$; $\sqrt{48}$.

EXERCISES

Simplify each of the following.

1. $\sqrt{24}$	2. $\sqrt{45}$	3. $\sqrt{32}$	4. $\sqrt{50}$
5. $\sqrt{90}$	6. $\sqrt{28}$	7. $\sqrt{75}$	8. $-\sqrt{200}$
9. $\sqrt{242}$	10. $\sqrt{18}$	11. $\sqrt{450}$	12. $\sqrt{10^3}$
13. $-\sqrt{32}$	14. $\sqrt{72}$	15. $\sqrt{250}$	16. $\sqrt{500}$

Multiplication Involving Radicals

You know that $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ where a and b are non-negative real numbers. Using this principle, we can perform multiplication involving radicals and we can express the products, using at most one radical sign. That is, $\sqrt{3} \cdot \sqrt{5} = \sqrt{15}$, and $\sqrt{\frac{1}{2}} \cdot \sqrt{6} = \sqrt{3}$. Similarly, if x and y represent non-negative real numbers, $\sqrt{\frac{1}{2}x} \cdot \sqrt{2y} = \sqrt{xy}$; $\sqrt{\frac{1}{3}x^3} \cdot \sqrt{6y} = \sqrt{2x^3y}$.

1. What does $\sqrt{\frac{1}{4}} \cdot \sqrt{8}$ equal? $\sqrt{\frac{2}{5}} \cdot \sqrt{15}$?

2. What does $\sqrt{6} \cdot \sqrt{\frac{2}{3}}$ equal? Can the radical you obtained be simplified further? What is the common name for this number?

3. The variables in the expressions below represent non-negative real numbers. Express each product, using at most one radical sign. Where a product is a perfect square, give the common name of the root.

a. $\sqrt{x}\cdot\sqrt{2y}$ **b.** $\sqrt{a}\cdot\sqrt{a}$ **c.** $\sqrt{y^2}\cdot\sqrt{z}$

d. $\sqrt{\frac{1}{2}a}\cdot\sqrt{2b}$ **e.** $\sqrt{6x}\cdot\sqrt{\frac{1}{3}y}$ **f.** $(\sqrt{a}\cdot\sqrt{b})\cdot(\sqrt{a})$

4. Find the products. Simplify wherever possible.

a. $\sqrt{5}\cdot\sqrt{\frac{1}{5}}$ **b.** $\sqrt{3}\cdot\sqrt{\frac{1}{3}}$ **c.** $\sqrt{\frac{1}{4}}\cdot\sqrt{4}$

d. $\sqrt{\frac{2}{7}}\cdot\sqrt{\frac{7}{2}}$ **e.** $\sqrt{\frac{3}{2}}\cdot\sqrt{\frac{2}{3}}$ **f.** $\sqrt{1\frac{1}{4}}\cdot\sqrt{\frac{4}{5}}$

In each case (**4a–f**), was the product equal to 1? Then $\sqrt{5}$ and $\sqrt{\frac{1}{5}}$, $\sqrt{3}$ and $\sqrt{\frac{1}{3}}$, and so on, are reciprocals or multiplicative inverses.

Look now at the *radicands* in **4a**. Are 5 and $\frac{1}{5}$ reciprocals? Is the same true in **4b**? **4c–e**? **4f**? Do you see that if the radicands of two radicals are reciprocals, then the radicals themselves are reciprocals? That is, $\sqrt{p}\cdot\sqrt{q}=1$ only if $p\cdot q=1$. (The variables p and q represent non-negative rational numbers.)

5. On page 263, it was stated that for any non-zero real number a there is a reciprocal or multiplicative inverse that can be denoted by $\frac{1}{a}$. Then can we express the reciprocal of $\sqrt{5}$ as $\frac{1}{\sqrt{5}}$? Use this form to express the reciprocal of $\sqrt{7}$; $\sqrt{11}$; $\sqrt{5^3}$; $\sqrt{a}$ $(a>0)$.

▶ For any positive real number a, the reciprocal of $\sqrt{a}$ can be represented as $\sqrt{\frac{1}{a}}$ or $\frac{1}{\sqrt{a}}$.

EXERCISES

Multiply. Express with at most one radical sign. If possible, express as an integer or a fraction.

1. $\sqrt{3}\cdot\sqrt{5}$ **2.** $-\sqrt{3}\cdot\sqrt{5}$ **3.** $(-\sqrt{3})\cdot(-\sqrt{3})$

4. $(-\sqrt{2})\cdot\sqrt{7}$ **5.** $\sqrt{8}\cdot\sqrt{\frac{1}{2}}$ **6.** $\sqrt{\frac{1}{2}}\cdot\sqrt{6}$

7. $(-\sqrt{7})\cdot(-\sqrt{8})$ **8.** $\sqrt{\frac{2}{5}}\cdot\sqrt{\frac{5}{2}}$ **9.** $\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{2}{3}}$

In the following, a and b represent non-negative real numbers.

10. $\sqrt{2a}\cdot\sqrt{3b}$ **11.** $\sqrt{6a}\cdot\sqrt{5b}$ **12.** $\sqrt{a}\cdot\sqrt{3}$

13. $\sqrt{a}\cdot\sqrt{b}$ **14.** $\sqrt{2b}\cdot\sqrt{2b}$ **15.** $\sqrt{3a}\cdot\sqrt{27a}$

16. What is the multiplicative inverse of $\sqrt{13}$? $\sqrt{\frac{1}{7}}$? $\sqrt{\frac{7}{3}}$? $\sqrt{3\frac{1}{2}}$?

Division Involving Radicals

Let us investigate division: $\sqrt{24}\div\sqrt{12}$. You know that to divide is the same as to multiply using the reciprocal of the divisor.

Then: $\sqrt{24}\div\sqrt{12}=\sqrt{24}\cdot\sqrt{\frac{1}{12}}$

$=\sqrt{\frac{24}{12}}$, or $\sqrt{2}$

In a similar way, $\sqrt{15}\div\sqrt{3}=\sqrt{15}\cdot\sqrt{\frac{1}{3}}$, or $\sqrt{5}$.

1. What does $\sqrt{32}\div\sqrt{2}$ equal? What is the common name for this number?

2. What does $\sqrt{6}\div\sqrt{3}$ equal?

Summarizing our steps, we have: $\sqrt{a}\div\sqrt{b}=\sqrt{a}\cdot\sqrt{\frac{1}{b}}$ $(b\neq 0)$.

But: $\sqrt{a}\cdot\sqrt{\frac{1}{b}}=\sqrt{a\cdot\frac{1}{b}}$, or $\sqrt{\frac{a}{b}}$

Thus, $\sqrt{a}\div\sqrt{b}=\sqrt{\frac{a}{b}}$

The division of $\sqrt{a}$ by $\sqrt{b}$ can be indicated by using the fraction bar, as $\frac{\sqrt{a}}{\sqrt{b}}$. Then:

▶ $\sqrt{a}\div\sqrt{b}$ $\left(\text{or }\frac{\sqrt{a}}{\sqrt{b}}\right)=\sqrt{\frac{a}{b}}$ where a and b are non-negative real numbers and $b\neq 0$.

3. In division examples involving a rational number and a radical, for example, $\sqrt{27}\div 3$ or $10\div\sqrt{2}$, we can express the rational number as a radical. Explain each step in these two illustrations.

$$\frac{10}{\sqrt{2}}=\frac{\sqrt{100}}{\sqrt{2}} \qquad \frac{\sqrt{27}}{3}=\frac{\sqrt{27}}{\sqrt{9}}$$
$$=\sqrt{50} \qquad =\sqrt{3}$$

Can $\sqrt{50}$ be simplified further? Perform this simplification.

EXERCISES

Divide.

1. $\sqrt{28} \div \sqrt{14}$
2. $\sqrt{3} \div \sqrt{3}$
3. $\sqrt{15} \div \sqrt{5}$
4. $\sqrt{48} \div \sqrt{16}$
5. $-\sqrt{6} \div \sqrt{3}$
6. $\sqrt{15} \div (-\sqrt{3})$
7. $-\sqrt{5} \div (-\sqrt{5})$
8. $-\sqrt{15} \div (-\sqrt{5})$
9. $-\sqrt{24} \div \sqrt{8}$
10. $\sqrt{24} \div (-\sqrt{12})$
11. $-\sqrt{6} \div (-\sqrt{3})$
12. $\sqrt{6} \div \sqrt{\frac{1}{2}}$
13. $\sqrt{6} \div \sqrt{\frac{1}{5}}$
14. $\sqrt{\frac{7}{3}} \div \sqrt{\frac{1}{3}}$
15. $\sqrt{5} \div \sqrt{\frac{5}{2}}$
16. $\frac{14}{\sqrt{14}}$
17. $\frac{\sqrt{5}}{5}$
18. $\frac{1}{2} \div \sqrt{2}$

Simplifying Products and Quotients Involving Radicals

1. Suppose you are asked to multiply $4\sqrt{15}$ by $2\sqrt{3}$. You know that this means $2 \cdot \sqrt{3} \cdot 4 \cdot \sqrt{15}$. Then, using the commutative and associative principles of multiplication:

$$2 \cdot \sqrt{3} \cdot 4 \cdot \sqrt{15} = 2 \cdot 4 \cdot \sqrt{3} \cdot \sqrt{15}$$
$$= 8\sqrt{45}$$

Simplify the radical $\sqrt{45}$. What does $8\sqrt{45}$ equal?

2. To perform the division $\sqrt{3} \div \sqrt{7}$, you might simply write $\frac{\sqrt{3}}{\sqrt{7}} = \sqrt{\frac{3}{7}}$. This answer seems reasonably simple since it has only one radical sign in it. However, for computational purposes, it is more convenient to have a radical in which the radicand is a whole number. Tables of approximate square roots give only the approximations for radicands that are whole numbers.

Of course, using a table, you could find that $\sqrt{3} \approx 1.732$ and $\sqrt{7} \approx 2.646$. (The sign $\approx$ means "is approximately equal to.") Then you could divide 1.732 by 2.646 to find an approximation of $\frac{\sqrt{3}}{\sqrt{7}}$.

On the other hand, would it not be easier to evaluate an expression such as $\frac{\sqrt{21}}{7}$ where no radical appears in the divisor? Using a table, you would find that $\sqrt{21} \approx 4.583$. Then you would simply divide 4.583 by 7.

Evaluate $\frac{\sqrt{3}}{\sqrt{7}}$, using the information given. Evaluate $\frac{\sqrt{21}}{7}$. Do $\frac{\sqrt{3}}{\sqrt{7}}$ and $\frac{\sqrt{21}}{7}$ appear to be names for the same number?

3. How can we show that $\sqrt{\frac{3}{7}}$ and $\frac{\sqrt{21}}{7}$ are equal? Consider the number $\sqrt{\frac{7}{7}}$. Is this another name for 1, the identity element for multiplication?

$$\text{Then: } \sqrt{\frac{3}{7}} = \sqrt{\frac{3}{7}} \cdot \sqrt{\frac{7}{7}}$$
$$= \sqrt{\frac{21}{49}}$$
$$= \frac{\sqrt{21}}{\sqrt{49}}, \text{ or } \frac{\sqrt{21}}{7}$$

Do you see why we chose $\sqrt{\frac{7}{7}}$ to name the identity element? We wished to make the divisor a perfect square. When we have an expression in which the radicand is not a fraction, we say we have *simplified* the expression.

4. Express each of the following in the form $\frac{\sqrt{a}}{b}$ where a and b are positive rational numbers.

a. $\frac{1}{\sqrt{2}}$ b. $\frac{2}{\sqrt{3}}$ c. $\frac{6}{\sqrt{5}}$ d. $\frac{2 \cdot \sqrt{14}}{\sqrt{7}}$

5. Perform the indicated operations. Simplify wherever possible.

a. $10\sqrt{3} \cdot 5\sqrt{5}$
b. $2(\sqrt{7} \cdot \sqrt{7})$
c. $5\sqrt{8} \cdot 3\sqrt{2}$
d. $3\sqrt{5} \cdot 2\sqrt{5}$
e. $6\sqrt{3} \div 2\sqrt{2}$
f. $5\sqrt{5} \div 10\sqrt{2}$
g. $\sqrt{7} \div \sqrt{14}$
h. $4\sqrt{6} \div 2\sqrt{2}$

6. The variables x and y represent positive rational numbers. Perform the indicated operations and simplify.

a. $x \div \sqrt{x}$
b. $\sqrt{x^3y} \div \sqrt{xy}$
c. $\sqrt{xy^2} \cdot \sqrt{x}$
d. $\sqrt{x^3y} \cdot \sqrt{xy}$

EXERCISES

Perform the indicated operations and simplify where possible.

1. $3\sqrt{8} \div 4\sqrt{2}$
2. $4\sqrt{3} \div \sqrt{3}$
3. $2\sqrt{2} \div \sqrt{24}$
4. $\frac{6\sqrt{15}}{3\sqrt{2}}$
5. $\frac{4\sqrt{7}}{8\sqrt{3}}$
6. $\frac{3\sqrt{2}}{6\sqrt{6}}$
7. $(-\sqrt{32}) \div 2\sqrt{2}$
8. $\sqrt{10^4} \div \sqrt{10^3}$
9. $\sqrt{10^3} \div \sqrt{10^4}$
10. $(\sqrt{10} \cdot \sqrt{3}) \div \sqrt{5}$
11. $(\sqrt{8} \cdot \sqrt{6}) \div \sqrt{2}$
12. $\sqrt{9} \div (\sqrt{3} \cdot \sqrt{6})$
13. $(\sqrt{15} \cdot \sqrt{6}) \div \sqrt{10}$
14. $(\sqrt{8})^2 \div 8$
15. $\sqrt{40} \div (\sqrt{5})^2$
16. $(\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{4}{3}}) \div \sqrt{3}$
17. $(-\sqrt{6})^2 \div \sqrt{3}$
18. $\sqrt{84} \div \sqrt{7}$

Adding and Subtracting Square Roots

1. What is another name for $\sqrt{9} + \sqrt{16}$? It is tempting to say this sum is $\sqrt{25}$, or 5. However, you know that $\sqrt{9} = 3$ and $\sqrt{16} = 4$. Therefore, an answer to the original question is 7, which is certainly not the same number as 5.

From this discussion, can you see that $\sqrt{a} + \sqrt{b}$ is not necessarily equal to $\sqrt{a + b}$?

▶ Therefore, to add $\sqrt{a}$ and $\sqrt{b}$, we simply indicate the sum: $\sqrt{a} + \sqrt{b}$.

2. Sometimes it is possible to express such a sum as a single term. For instance, consider $\sqrt{12} + \sqrt{27}$. Does $\sqrt{12}$ equal $\sqrt{4}\cdot\sqrt{3}$, which is simply $2\sqrt{3}$? Does $\sqrt{27}$ equal $3\sqrt{3}$? Explain.

Then $\sqrt{12} + \sqrt{27} = 2\sqrt{3} + 3\sqrt{3}$. By what principle may we express this as $(2 + 3)\sqrt{3}$? Thus, $\sqrt{12} + \sqrt{27} = 5\sqrt{3}$.

Let us see what made it possible to express the sum $\sqrt{12} + \sqrt{27}$ as a single term. Do you see that each of the original terms—$\sqrt{12}$ and $\sqrt{27}$—could be expressed as the product of some rational number and the same irrational number $(\sqrt{3})$? Then we had an expression of the form $ac + bc$, which you know can be written as $(a + b)c$. That is, $k_1\sqrt{x} + k_2\sqrt{x} = (k_1 + k_2)\sqrt{x}$. Remember, if we are to apply the distributive principle in such cases, the same radical must appear in each term.

If possible, express as a single term.

	a	b	c
3.	$3\sqrt{2} + 4\sqrt{2}$	$5\sqrt{3} - 2\sqrt{3}$	$4\sqrt{7} + \sqrt{7}$
4.	$\sqrt{18} + \sqrt{2}$	$\sqrt{45} - \sqrt{20}$	$3\sqrt{12} + \sqrt{18}$
5.	$3\sqrt{3} + \sqrt{48}$	$2\sqrt{50} - 5\sqrt{18}$	$\sqrt{2} - \sqrt{8}$
6.	$2\sqrt{2} - \sqrt{8}$	$\frac{1}{2}\sqrt{12} + \frac{1}{3}\sqrt{27}$	$\frac{1}{2}\sqrt{32} - \frac{1}{5}\sqrt{50}$
7.	$2\sqrt{150} - \frac{3}{8}\sqrt{96}$	$\frac{2}{3}\sqrt{5} - \frac{1}{3}\sqrt{20}$	$.3\sqrt{75} - .2\sqrt{300}$

Express in as few terms as possible.

8. $2\sqrt{8} - \frac{1}{2}\sqrt{12} - 3\sqrt{18} + 2\sqrt{27}$
9. $10\sqrt{\frac{1}{5}} + \sqrt{72} - 10\sqrt{\frac{1}{2}} - \sqrt{20}$
10. $3\sqrt{18} + \sqrt{98} - 5\sqrt{48} - 6\sqrt{\frac{1}{3}}$

Combining Operations

1. Does $\sqrt{2}(\sqrt{3} + \sqrt{5})$ look like $a(b + c)$ where a, b, and c represent real numbers? Then the distributive principle should tell us how to multiply $(\sqrt{3} + \sqrt{5})$ by $\sqrt{2}$. Complete the multiplication. Do you obtain $\sqrt{6} + \sqrt{10}$?

Multiply and simplify.

	a	b	c
2.	$\sqrt{3}(2\sqrt{2} - \sqrt{5})$	$2\sqrt{3}(\sqrt{5} - \sqrt{7})$	$3\sqrt{2}(2\sqrt{2} - 1)$
3.	$(\sqrt{7} + \sqrt{3})\sqrt{2}$	$-\sqrt{3}(\sqrt{2} + 3)$	$2\sqrt{5}(3\sqrt{6} - \sqrt{3})$
4.	$3\sqrt{a}(\sqrt{a} + \sqrt{c})$	$\sqrt{3}(2\sqrt{5} - 3\sqrt{7})$	$\sqrt{ab}(\sqrt{a} + \sqrt{b})$
5.	$\sqrt{3}(\sqrt{6} - \sqrt{3})$	$2\sqrt{2}(3\sqrt{6} - \sqrt{5})$	$3\sqrt{6}(2\sqrt{12} - 3\sqrt{7})$

6. To divide $\sqrt{12} + \sqrt{18}$ by $\sqrt{2}$ is to multiply $\sqrt{12} + \sqrt{18}$ by $\frac{?}{}$. You might answer $\frac{1}{\sqrt{2}}$ or $\sqrt{\frac{1}{2}}$. Try the multiplication either way. Do you obtain $\sqrt{6} + 3$?

7. Simplify.

a. $\frac{\sqrt{24} + \sqrt{54}}{\sqrt{3}}$ b. $\frac{12\sqrt{10} - 8\sqrt{15}}{2\sqrt{5}}$ c. $\frac{6\sqrt{28} - 18\sqrt{14}}{3\sqrt{7}}$

8. Is an expression of the form $\frac{x + y}{z}$ $(z \neq 0)$ equivalent to $\frac{x}{z} + \frac{y}{z}$? Explain your answer fully, using the fact that division is defined in terms of multiplication.

To perform the division $\frac{6 + \sqrt{12}}{2}$, you can divide 6 by 2 and $\sqrt{12}$ by 2. The first quotient is $\frac{?}{}$. The second quotient can be found by replacing 2 by $\sqrt{\frac{?}{}}$. Complete the division. Do you obtain $3 + \sqrt{3}$?

9. Simplify.

a. $\frac{12 + \sqrt{24}}{2}$ b. $\frac{24 - \sqrt{108}}{6}$ c. $\frac{18 + \sqrt{80}}{4}$

EXERCISES

1. State the index of each of the following: $\sqrt{18}$, $\sqrt[3]{64}$, $\sqrt[5]{\frac{1}{5}}$, $\sqrt[7]{2}$, $\sqrt{7}$. What is the radicand of each as it now stands?

Perform the indicated operations and simplify where possible.

	a	b	c
2.	$5\sqrt{2}\cdot 6\sqrt{3}$	$2\sqrt{5}\cdot 10\sqrt{6}$	$(-4\sqrt{2})(5\sqrt{5})$
3.	$(-10\sqrt{5})(-5\sqrt{10})$	$3\sqrt{\frac{1}{2}}\cdot 4\sqrt{6}$	$\frac{1}{2}\sqrt{5}\cdot 6\sqrt{\frac{2}{5}}$
4.	$(3\sqrt{2})^2$	$(4\sqrt{3})^2$	$(-3\sqrt{5})^2$
5.	$3\sqrt{5} \div 4\sqrt{5}$	$(-10\sqrt{7})(-3\sqrt{7})$	$(3\sqrt{2})^2(4\sqrt{3})$
6.	$2\sqrt{.5} \div \sqrt{.5}$	$\left(\sqrt{\frac{2}{3}}\right)^2\cdot\left(\sqrt{\frac{3}{2}}\right)^2$	$(3\sqrt{2})^3$
7.	$(3\sqrt{3})^3$	$(\frac{1}{3}\sqrt{2})^3$	$(\sqrt{2})^3\cdot(\sqrt{3})^3$
8.	$(\sqrt{2})^4$	$(\sqrt{3})^4 \div \sqrt{3}$	$(2\sqrt{2})^3 \div (3\sqrt{2})^2$

Express each of the following as a radical with coefficient 1. For instance, $2\sqrt{3} = \sqrt{4}\cdot\sqrt{3}$, or $\sqrt{12}$.

9.	$2\sqrt{5}$	$3\sqrt{2}$	$3\sqrt{3}$
10.	$\frac{1}{2}\sqrt{7}$	$2\sqrt[3]{2}$	$2\sqrt[3]{3}$

Assume that the variables below are positive rational numbers. Multiply and simplify.

11.	$3\sqrt{a}\cdot 2\sqrt{2a}$	$x\sqrt{y}\cdot y\sqrt{x}$	$\sqrt{x^2y}\cdot\sqrt{y}$
12.	$2\sqrt{a^4b}\cdot\sqrt{b}$	$(-\sqrt{b^5})\cdot(\sqrt{b^3})$	$(-\frac{1}{2}\sqrt{a})(-\frac{1}{2}\sqrt{a})$

Combine into as few terms as possible.

13. a. $2\sqrt{a} + 5\sqrt{a} - 3\sqrt{a}$ **b.** $4\sqrt{b^3} + 2b\sqrt{9b} - \frac{2}{b}\sqrt{16b^5}$

14. a. $4\sqrt{\frac{a}{2}} - \frac{1}{2}\sqrt{8a} + \sqrt{50a}$ **b.** $\sqrt{\frac{a}{2}} + a\sqrt{\frac{2}{a}}$

Using the fact that $\sqrt{2} \approx 1.414$, $\sqrt{3} \approx 1.732$, and $\sqrt{5} \approx 2.236$, find approximations for each of the irrational numbers named below.

	a	b	c
15.	$\sqrt{18}$	$\sqrt{500}$	$\sqrt{180}$
16.	$\sqrt{48}$	$\sqrt{75}$	$\sqrt{125}$

USING THE REAL NUMBERS

Radical Equations

1. Is the sentence $\sqrt{2x+3} = 3$ true if x is replaced by 0? by 1? by 2? by 3? Do you think that the solution set contains only 3?

To answer this question and to solve equations in which the variable is part of a radicand, we shall look for a systematic method of solution.

2. Suppose two numerals are names for the same number; for instance, $\sqrt{6+3}$ and $\sqrt{8+1}$. Is it true then that $(\sqrt{6+3})^2 = (\sqrt{8+1})^2$? This is a basic principle in solving equations with square roots. Stated briefly: If $a = b$, then $a^2 = b^2$.

3. Now let us reverse the question and ask: If a^2 and b^2 name the same number, is it necessarily true that $a = b$? Consider the following: $(-4)^2 = 4^2$ is a true statement. Does it follow then that $-4 = 4$?

From the discussion in **2** and **3**, we can conclude: If we use the operation of "squaring" in solving an equation, the solution we find will be the *possible* solution of the original equation but it will not *necessarily* be the solution.

4. Let us see how the equation $\sqrt{2x+3} = 3$ can be solved systematically. We wish to eliminate the radical, so we square each member, as shown in line (2). We then proceed in the way that is familiar to you.

How was the equation in line (3) obtained? the equation in line (4)? The solution set of the last equation is {3}. Does 3 satisfy the original equation? Then the solution set of $\sqrt{2x+3} = 3$ is {3}.

(1)	$\sqrt{2x+3} = 3$
(2)	$(\sqrt{2x+3})^2 = 3^2$
	or $2x + 3 = 9$
(3)	$2x = 6$
(4)	$x = 3$

5. As another illustration, study the work for $\sqrt{5y} - 4 = 1$. We begin by writing an equivalent equation in which the radical stands alone in one member. Then we square and proceed as shown. What is the root of the last equation? Does this number satisfy the original equation? That is, is $\sqrt{5\cdot 5} - 4 = 1$ a true statement?

(1)	$\sqrt{5y} - 4 = 1$
(2)	$\sqrt{5y} = 5$
(3)	$(\sqrt{5y})^2 = 5^2$
	or $5y = 25$
(4)	$y = 5$

6. Perhaps you wonder why we wanted to have the radical by itself in one member. Suppose we had begun by squaring the members of the original equation $\sqrt{5y} - 4 = 1$. $(\sqrt{5y} - 4)^2$ means $(\sqrt{5y} - 4)(\sqrt{5y} - 4)$. Will the left member of the new equation be $5y - 8\sqrt{5y} + 16$? Explain. You can see that this expression is more complicated than the original one.

7. Another solution of a radical equation appears at the right. What is the root of the last equation? Then 4 is a possible root of the original equation. We must, however, check to see whether 4 is actually a root.

Is $3 + 7 = 4$ a true statement?

Then 4 *is not* a root of the original equation. The solution set of $\sqrt{x + 5} + 7 = 4$ is $\emptyset$, the empty set.

$$\sqrt{x + 5} + 7 = 4$$
$$\sqrt{x + 5} = -3$$
$$x + 5 = 9$$
$$x = 4$$

Check:

$$\sqrt{4 + 5} + 7 = 4$$
$$\sqrt{9} + 7 = 4$$
$$3 + 7 = 4$$

EXERCISES

Solve the following equations. Check to be sure the members of your solution set really satisfy the given equation.

1. $\sqrt{3x} = 2$

2. $\sqrt{x + 3} = 2$

3. $\sqrt{5x} = \frac{1}{2}$

4. $\sqrt{x - 5} = 3$

5. $\sqrt{\frac{x}{2}} = -3$

6. $\sqrt{\frac{x}{3}} = 4$

7. $\sqrt{\frac{2x}{3}} = 2$

8. $\sqrt{\frac{3x}{5}} = 6$

9. $\sqrt{2x + 3} = 5$

10. $\sqrt{2x - 3} = 5$

11. $\sqrt{2x + 4} = 1$

12. $2\sqrt{x} = 3$

13. $\frac{1}{2}\sqrt{x + 2} = 2$

14. $2\sqrt{3n + 1} = 1$

15. $\sqrt{3x - 2} - 7 = 1$

16. $\sqrt{\frac{2x - 4}{5}} = 2$

17. Look back at the equation in Exercise **5**. Do you see how you might have decided at once that the solution set was $\emptyset$? If not, answer these questions: By definition, must $\sqrt{\frac{x}{2}}$ represent a non-negative number? Then could $\sqrt{\frac{x}{2}} = -3$ ever be a true statement?

18. Must the solution set of $\sqrt{x + 3} + 5 = 1$ be $\emptyset$? Explain your answer.

Radical Formulas

1. You know the formula for the area of a square is $A = s^2$. Must s be a positive number? Must A also represent a positive number? If we take the positive square root of both members of the formula $A = s^2$, then we may write: $s = \sqrt{A}$. This is an example of a formula having a radical. Find s if A is replaced by 16; by 144. Approximate s to the nearest tenth if the area of a square is 48 square inches.

2. The formula that tells how many seconds, t, it takes an object to fall a distance of s feet is $t = \frac{\sqrt{s}}{4}$. Find the number of seconds it takes an object to fall 64 feet; 100 feet. Approximate the number of seconds, to the nearest tenth of a second, if the object falls 108 feet; 160 feet.

3. If a balloon is h feet above the surface of the earth, the number of miles, m, from which it may be seen is given by the formula $m = \sqrt{\frac{3h}{2}}$. If the balloon is 600 feet above the earth, from how far away can it be seen? if it is 2400 feet above the earth? Approximate the distance to the nearest tenth of a mile if the balloon is 2000 feet above the earth; 10,000 feet above the earth.

4. The formula $v = \sqrt{2000p}$ tells the velocity, v, in miles per hour, of a plane, if the air pressure on the wings is p pounds per square foot. What is the velocity when the pressure is 20 pounds? Approximate the velocity to the nearest tenth of a mile when the pressure is 40 pounds.

Transforming Formulas

In formulas relating to area, volume, and so on, variables must be assumed to represent positive numbers. In all of the work that follows, this assumption is made.

1. This formula, made famous by Einstein, relates energy (E), mass (m), and the speed of light (c). To transform the formula so that c is the subject, we begin by multiplying each member by $\frac{1}{m}$. Then we take the positive square root of each member. Can we express the formula as $c = \sqrt{\frac{E}{m}}$?

$$E = mc^2$$
$$\frac{E}{m} = c^2$$

2. Transform the formula $A = \pi r^2$ so that the subject of the formula is r; that is, so that r is by itself as one member of the equation.

Transform each of the formulas in **3–8** as indicated.

3. $S = 4\pi r^2$. This is the formula for the area, S, of a sphere whose radius is r. Make r the subject.

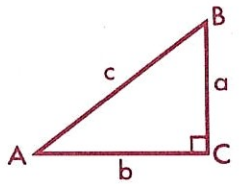
4. $S = 6e^2$. This is the formula for the area, S, of a cube whose edge is e. Make e the subject.

5. $V = \pi r^2 h$. This is the formula for the volume, V, of a cylinder whose height is h and whose base radius is r. Make r the subject.

6. $H = \frac{1}{6}D^2N$. This is the formula for finding the horsepower, H, of a certain engine where D is the diameter of a cylinder and N is the number of cylinders. Make D the subject.

7. $W = I^2R$. In this formula, W is the power in watts, I the current in amperes, and R the resistance in ohms. Make I the subject.

8. $p = .003v^2$. This formula tells how to find the pressure, p, in pounds per square foot, of a wind blowing with the speed of v miles per hour. Make v the subject.



Pythagorean Rule

From your previous courses in mathematics, you are familiar with the **Pythagorean rule**, sometimes referred to as the **right triangle rule.** The rule can be stated as follows:

▶ The sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse.

The diagram above shows a right triangle ABC, with the customary notation for the lengths of the sides of the triangle. That is, the length of the side opposite vertex A is designated by a, the length of the side opposite vertex B is designated by b, and the length of the hypotenuse (which is opposite the right angle C) is designated by c. In symbols, then, the Pythagorean rule can be stated as follows:

▶ In right triangle ABC, $a^2 + b^2 = c^2$.

1. Suppose a is 3 centimeters, and b is 4 centimeters, and you wish to determine c. What does a^2 equal? b^2? Then does c^2 equal 25? Since lengths are positive, what is the value of c?

2. Suppose a is 5 inches and b is 12 inches. What is the length of the hypotenuse?

3. Suppose a is 3 centimeters and b is 6 centimeters. What length is c? Do you obtain $\sqrt{45}$ centimeters as this length? Simplify the radical, but leave it in radical form.

The irrational number $\sqrt{45}$, or $3\sqrt{5}$, is *exact*; if we use a decimal representation of this number, we must remember that this is simply an approximation.

4. If a is $\sqrt{3}$ inches and b is 3 inches, what does c equal? Simplify the radical.

5. Find c in each of these situations.

$a = 4$ in., $b = 6$ in.	$a = 2$ cm, $b = 3$ cm
$a = 5$ in., $b = 4$ in.	$a = 1$ cm, $b = 7$ cm

Do you see that in many cases irrational numbers are necessary to name lengths exactly?

6. By taking the positive square root of the members of the formula $a^2 + b^2 = c^2$, do you find that $c = \sqrt{a^2 + b^2}$? This form indicates how you computed c in **1–5.** You found the sum of a^2 and b^2 and then named the positive number that was the square root of the sum $a^2 + b^2$.

7. Sometimes we know the length of one leg of a right triangle and the length of the hypotenuse of the right triangle and we wish to find the length of the unknown leg. To do so, we may make use of other forms of the Pythagorean rule.

$$a = \sqrt{c^2 - b^2} \qquad b = \sqrt{c^2 - a^2}$$

Transform the Pythagorean rule to obtain each of these forms of it. Show all your work.

8. Find a if c is 41 yards and b is 9 yards.

9. Find c if a is 35 feet and b is 12 feet.

10. Find b if c is 29 yards and a is 20 yards.

Problem Solving

1. How long is the side of a square whose area is 30 square inches? Leave answer as a radical.

2. What is the radius of a circle whose area is 144π square feet?

3. A number is diminished by 3. The positive square root of the difference is 11. What is the number?

4. The positive square root of the sum of two consecutive integers is 7. Does the equation $\sqrt{x + (x + 1)} = 7$ express the stated condition? Solve the equation and find the two integers.

5. The positive square root of the sum of three consecutive even integers is 6. Find the three numbers.

6. The positive square root of the sum of three consecutive odd integers is 3. What are the three integers?

7. The negative square root of the sum of four consecutive odd integers is -4. What are the four integers?

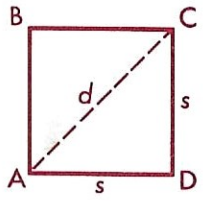
8. The hypotenuse of a right triangle is 9 yards long, and one leg of the triangle is 7 yards long. What is the length of the other leg?

9. Are there three consecutive integers such that the square root of their sum is 6? If so, name them.

10. Are there three consecutive integers such that the square root of their sum is 7?

11. Each of the legs of a right triangle is $\sqrt{8}$ feet long. How long is the hypotenuse?

12. A horse is to be tied to a stake and permitted to graze over 300 square feet. Find the length of the rope from the stake to the horse, to the nearest foot. (Use $A = \pi r^2$.)



On Your Own

1. As you know, two right triangles are formed when a diagonal of a square is drawn. In this diagram, the length of the diagonal is denoted by d; the length of a side of the square is s.

a. Find d when s is 2 centimeters. Simplify the radical.
b. Find d when s is 3 centimeters. Simplify the radical.
c. Find d when s is 4 centimeters. Simplify the radical.

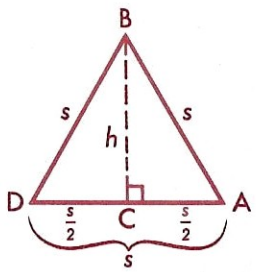
2. Look for a pattern in your answers. When s is 5 centimeters, what do you think d equals? when s is 6 centimeters? For any allowable value of s, $d = s \cdot \underline{?}$.

3. Using the tools of algebra, you can develop the formula yourself. In any square $ABCD$, if s is the length of a side and d is the length of the diagonal, is the following true: $d^2 = s^2 + s^2$? How can you express $s^2 + s^2$ more simply? Then $d^2 = 2s^2$, and $d = \sqrt{2s^2}$. Complete the work.

4. The altitude in any equilateral triangle bisects the base. This fact is indicated on the diagram. Suppose we wish a formula for the length of the altitude. We write:

$$h^2 + \left(\frac{s}{2}\right)^2 = s^2$$

Express this formula with h as the subject.



5. Using the formula $A = \frac{1}{2}bh$ and the results obtained in **4**, prove that the formula for the area of any equilateral triangle is given by $A = \frac{s^2\sqrt{3}}{4}$.

RELATING RATIONAL AND IRRATIONAL NUMBERS

Suppose you are told that $\frac{1}{2}$ of a number x is equal to $\sqrt{5}$. (That is, $\frac{1}{2}x = \sqrt{5}$.) Then is x equal to $2\sqrt{5}$?

Is $2\sqrt{5}$ a rational number, or is it an irrational number? The answer to this question is easily found by thinking of 2 as $\sqrt{4}$. Then $2\sqrt{5} = \sqrt{4} \cdot \sqrt{5}$, or $\sqrt{20}$. 20 is not a perfect square; therefore, by the principle stated on page 267, $\sqrt{20}$ or $2\sqrt{5}$ is an irrational number.

Throughout your work with radicals, you have been obtaining expressions such as $2\sqrt{5}$ or $4 + \sqrt{7}$ as answers to problems. We have not spoken of whether these expressions are rational numbers or irrational numbers. Let us investigate such questions now. Then, as you proceed in future chapters, you will be able to recognize whether the root of an equation is a rational number or an irrational number.

Products

If you multiply any irrational number by 2, do you think the result will be rational, or irrational? Since the irrational number is a non-repeating decimal, it seems reasonable that the product will also be non-repeating, but let us see if we can prove that the result must be an irrational number.

Let v be the irrational number. How would you write the product of v and 2? We know that $2v$ is either a rational number or an irrational number. Let us investigate the results if we *assume* that $2v$ is a rational number.

If $2v$ is a rational number, then we can express it as the quotient of two integers. We write:

$$2v = \frac{P}{Q} \quad \text{where } P \text{ and } Q \text{ are integers, } Q \neq 0$$

Then $\frac{1}{2} \cdot 2v = \frac{1}{2} \cdot \frac{P}{Q}$

or: $v = \frac{P}{2Q}$

Since Q is an integer, must 2Q also be an integer? Since P is an integer and 2Q is an integer, then v is equal to the quotient of two integers. Does this make v a rational number?

But this contradicts the fact from which we started—namely, v is irrational. Thus, our assumption that "$2v$ is a rational number" has led to a contradiction. Therefore, we can conclude that $2v$ is irrational.

1. Show that if v is an irrational number, so is $3v$. (Use a procedure similar to the procedure shown above.)

2. Show that the number that is seven times an irrational number is an irrational number.

3. Show that the product of any integer S and any irrational number v is an irrational number.

4. Show that the product of any irrational number v and the rational number $\frac{3}{4}$ is an irrational number.

5. Show that the product of any irrational number v and any non-zero rational number $\frac{S}{T}$ is an irrational number.

Sums

If you add 2 to an irrational number, is the sum a rational number, or an irrational number? You recall that an irrational number is represented by a non-terminating, non-repeating decimal. For example, 36.101001000100001 . . . is an irrational number (since the decimal is non-terminating and non-repeating). At the right, we show the result of adding 2 to this irrational number. You can see that no change occurs in the non-terminating, non-repeating character of the decimal. Is the sum irrational?

$$\begin{array}{r} 36.101001000100001\ \ldots \\ 2 \\ \hline 38.101001000100001\ \ldots \end{array}$$

In a similar way, it can be shown that if we add any integer to an irrational number we obtain a sum that is an irrational number.

1. If you add $\frac{2}{3}$ to an irrational number, do you think that the sum is rational? To see that the sum is irrational, let v represent the original (irrational) number. Is $\frac{3v}{3}$ another way of expressing v? Then we can proceed as shown.

Notice what has been done. Using familiar principles, we have re-expressed $v + \frac{2}{3}$ as the product of $\frac{1}{3}$ and the expression $3v + 2$. If we can prove that this product is an irrational number, then we will know that $v + \frac{2}{3}$ is an irrational number.

$$\begin{aligned} v + \tfrac{2}{3} &= \tfrac{3v}{3} + \tfrac{2}{3} \\ &= \tfrac{3v + 2}{3} \\ &= \tfrac{1}{3} \cdot (3v + 2) \end{aligned}$$

Is $\frac{1}{3}$ a rational number? Consider next the expression $3v + 2$. Since v is irrational, is $3v$ irrational? Since $3v$ is irrational, $3v + 2$ is also irrational. Why?

Thus, $\frac{1}{3}(3v + 2)$ represents the product of a rational number and an irrational number, and you know that this product must be irrational. Therefore, $v + \frac{2}{3}$ is an irrational number.

2. Show that if v is an irrational number, $\frac{5}{7} + v$ is an irrational number.

3. Show that we obtain an irrational number as the sum of any irrational number v and any rational number $\frac{P}{Q}$ (P and Q are integers, and $Q \neq 0$).

▶ The product or the sum of a rational number and an irrational number is an irrational number. Thus, any number such as $2\sqrt{5}$, $3\sqrt{11}$, $4 + \sqrt{7}$, or $1 + 3\sqrt{13}$ is an irrational number.

EXERCISES

1. Consider the product of two irrational numbers.

   a. Are $\sqrt{5}$ and $\sqrt{3}$ irrational numbers? Is their product also an irrational number?

   b. Are $\sqrt{2}$ and $\sqrt{8}$ irrational numbers? Is $\sqrt{2}\cdot\sqrt{8}$ an irrational number?

   c. Is the product of two irrational numbers always an irrational number? always a rational number? sometimes irrational and sometimes rational?

2. Consider next the sum of two irrational numbers.

   a. Are the numbers .1010010001 . . . , .2020020002 . . . , and .3030030003 . . . all irrational numbers? Is the third number the sum of the first two? In this case, is the sum of two irrational numbers an irrational number?

   b. If v is an irrational number, is $-v$ an irrational number? Is $2 + v$ also an irrational number? Consider the sum of $2 + v$ and $-v$.

$$\begin{aligned}(2 + v) + (-v) &= 2 + [v + (-v)] \\ &= 2 + 0 \\ &= 2\end{aligned}$$

   Then, in this case, is the sum of two irrational numbers a rational number?

   c. Is the sum of two irrational numbers always a rational number? always an irrational number? sometimes rational and sometimes irrational?

3. Complete each of the following statements with one of these choices: *rational, irrational, sometimes rational and sometimes irrational.* For **a** and **b**, you will have to recall what you learned in Chapter 2.

   a. The product of two rational numbers is ___?___.
   b. The sum of two rational numbers is ___?___.
   c. The product of a rational and an irrational number is ___?___.
   d. The sum of a rational and an irrational number is ___?___.
   e. The product of two irrational numbers is ___?___.
   f. The sum of two irrational numbers is ___?___.

1. Express each of the following as a terminating or repeating decimal. 258–259

   a. $\frac{3}{8}$ b. $\frac{7}{11}$ c. $\frac{5}{3}$ d. $\frac{53}{99}$ e. $\frac{4}{15}$

2. Express each of the following as the indicated quotient of two integers. 260

   a. .78 b. $.\overline{78}$ c. $2.\overline{7}$ d. .35353535 . . . e. $.\overline{3}$

3. Give three illustrations of non-repeating, non-terminating decimals. Are these rational numbers? 261–262

4. Arrange the following real numbers in order of size, starting with the smallest. 264

   a. $5.\overline{343}$ b. $-.\overline{34}$ c. .34343433433334333334 . . .
   d. $.\overline{34}$ e. 0 f. −.34343433433334333334 . . .

5. Approximate the following to the nearest hundredth. 265

   a. $-\sqrt{3}$ b. $\sqrt{59}$ c. $\sqrt{300}$ d. $\sqrt{153}$

6. Rewrite each of the following in simplified form, using at most one radical sign. 271–273

   a. $\sqrt{50}\cdot\sqrt{2}$ b. $-\sqrt{8}\cdot\sqrt{18}$ c. $\sqrt{6} \div \sqrt{3}$
   d. $\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{3}{2}}$ e. $\frac{\sqrt{80}}{\sqrt{40}}$ f. $\sqrt{\frac{7}{8}} \div \sqrt{\frac{1}{32}}$
   g. $\frac{3}{4}\sqrt{2}\cdot 4\sqrt{8}$ h. $2\sqrt{12} \div 4\sqrt{3}$ i. $15\sqrt{72} \div 9\sqrt{2}$

7. Knowing $\sqrt{2} \approx 1.414$, approximate: $\sqrt{\frac{1}{2}}$; $\sqrt{\frac{49}{2}}$. 274

8. Express as a single term: $\sqrt{18} + \sqrt{2}$. 276

9. Simplify: $2\sqrt{5}\,(3\sqrt{5} - 4\sqrt{2})$; $\frac{\sqrt{24} + 2\sqrt{96}}{\sqrt{3}}$. 277

10. Find the solution set of each of the following. 280

    a. $\sqrt{5x} = 7$ b. $\sqrt{2x - 5} + 4 = 3$ c. $\sqrt{3x + 1} - 2 = 2$

11. The square root of the sum of three consecutive integers is 12. What are the integers? 284

If you had difficulty with any of the work on this test, find help on the pages listed at the ends of the rows.

Extra Practice Page 406

Challenge Page 421

TO BE CONTINUED

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